Homework #1 — PHYS 625 — Spring 2021 Professor Victor Galitski
Office: 2270 PSC

Deadline: Monday, February 8, 2020, by email to masoudma@umd.edu before class

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Do not forget to write your name and the homework number!

Bogoliubov transformation

1. Consider a classical chain of oscillators (see also your lecture notes)

$$\mathcal{H} = \sum_{n=-\infty}^{+\infty} \left[\frac{p_n^2}{2m_n} + \frac{K}{2}(x_{n+1} - x_n)^2 \right] \tag{1}$$

where $K = m\omega_0^2$, $m_n = m$ if n is even and $m_n = M > m$, if n is odd. Find the speed of sound in the chain, using the Laplace formula

$$c^2 = \frac{\partial P}{\partial \rho}$$

where P is the pressure and ρ is the density. Note that in 1D, pressure and force are the same thing. Compare your result with that obtained in the first lecture.

2. Consider a quantum chain of oscillators (see also your lecture notes)

$$\mathcal{H} = \sum_{n=-\infty}^{+\infty} \left[\frac{\hat{p}_n^2}{2m_n} + \frac{m\omega_0^2}{2}(\hat{x}_{n+1} - \hat{x}_n)^2 \right] \tag{2}$$

Diagonalize the Hamiltonian by following the process, described in lectures 2 and 3, but filling all the gaps, missing steps, and reproducing all the algebra and technical details. Specifically

(a) Rewrite the Hamiltonian in terms of local creation/annihilation operators $\hat{a}_n^\dagger$ and $\hat{a}_n$.

(b) Perform a Fourier transform $\hat{a}_n = \int_q \hat{\tilde{a}}_q e^{iqn}$ and rewrite the Hamiltonian in terms of $\hat{\tilde{a}}_q^\dagger$ and $\hat{\tilde{a}}_q$ (you can drop "~" for the sake of brevity, like was done in class).

(c) Perform the Bogoliubov transformation $\hat{a}_q = u_q\hat{b}_q + v_q\hat{b}_q^\dagger$, represent the Bogoliubov amplitudes u_q and v_q in terms of hyperbolic functions (see lectures), and once again write the Hamiltonian in terms of $\hat{b}_q$ and $\hat{b}_q^\dagger$.

(d) Find the Bogoliubov amplitudes that correspond to a diagonal Hamiltonian in terms of b's.

(e) Reproduce the spectrum of the collective modes (acoustic phonon).

3. Consider a quantum chain of oscillators) described by the Hamiltonian 2 above. Diagonalize the Hamiltonian by quantizing the classical normal modes.

I.e., do the following: First, following your lecture notes from the first lecture, solve the classical problem (with $m = M$) and find the classical normal modes (do not simply take the limit of $m \to M$ in the more general problem we solved, but repeat all calculations in the simpler case of one type of atom in an elementary cell). Associate with each classical normal mode an appropriate set of quantum operators and write down the "diagonal" Hamiltonian (and associated commutation relations of the operators involved). Compare your result with the one obtained using the Bogoliubov transform.