Homework #2 — PHYS 625 — Spring 2021
Professor Victor Galitski
Office: 2270 PSC

Deadline: Monday, February 22, 2020, by email to masoudma@umd.edu before class

TA: Masoud Arzanagh
masoudma@umd.edu

Web page: https://terpconnect.umd.edu/ galitski/PHYS625/index.html

Do not forget to write your name and the homework number!

Bogoliubov transformation for bosons & fermions; bands in a periodic potential

1. Consider a system of interacting bosons, described by the Hamiltonian,

$$\hat{\mathcal{H}} = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{U}{2V} \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4} \hat{a}^\dagger_{\mathbf{p}_4} \hat{a}^\dagger_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_2} \hat{a}_{\mathbf{p}_1} \tag{1}$$

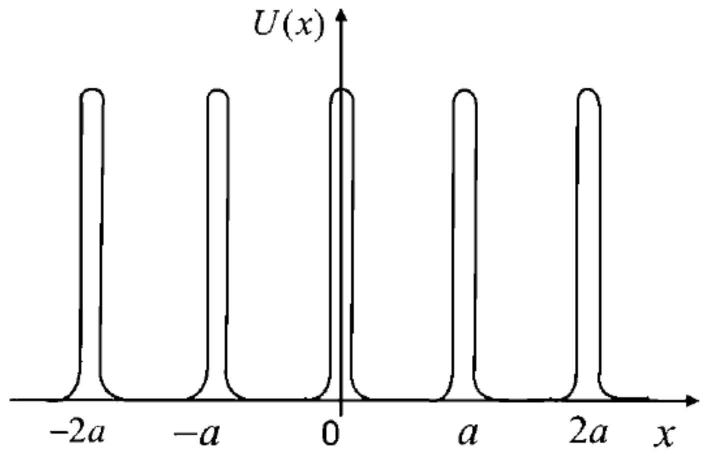
where $\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}$ are creation and annihilation operators of bosons with momentum $\mathbf{p}$ correspondingly, U is the interaction constant, and V is the volume of the system. Here the same notations (and summation over momenta instead of an integration; this is unimportant) are used as in the recommended book by Abrikosov, Gor'kov, and Dzyaloshinskii. You are advised to follow the book, but present details of all calculations.

Use the Bogoliubov approximation (where the operators corresponding to the zero-momentum of the condensate $\hat{a}_{\mathbf{0}}$ are replaced by numbers of order $\sqrt{N_0}$; N_0 is the number of particles in the condensate, that is assumed to much exceed the number particles outside the condensate, $N_0 \gg N - N_0$) to derive a quadratic-in-operators Hamiltonian. Use the Bogoliubov transform to diagonalize the Hamiltonian and find the spectrum of excitations in the BEC. Calculate the speed of sound.

2. Using Bogoliubov transformations, diagonalize the following **fermion** Hamiltonian ($J_{1,2}$ and B are some constants):

$$\hat{\mathcal{H}} = \sum_{n=-\infty}^{+\infty} \left[J_1 \hat{f}^\dagger_n \hat{f}_{n+1} + J_2 \hat{f}_n \hat{f}_{n+1} - B \hat{f}^\dagger_n \hat{f}_n + \text{H. c.} \right]$$

This Hamiltonian appears in the context of a one-dimensional quantum magnetic (specifically the XY-model, as discussed in class). Find the spectrum of quasiparticles, $\varepsilon(k)$ of this Hamiltonian. Note that for $J_1 = J_2$ and $B = 0$ the dispersion disappears. Can this fact be understood without calculations?



3. Consider a quantum particle of mass m in the 1D periodic potential of the form,

$$U(x) = \alpha \sum_{n=-\infty}^{\infty} \delta(x - na).$$

This potential can be viewed as a toy model of an ideal one-dimensional "crystal" (see Figure above),

Find a system of independent solutions of the Schrödinger equation for an arbitrary value of the energy E and determine the energy spectrum.

You may (but are not required to) follow the steps below

- Write a plane-wave form of the solution in the region, $n < x/a < (n+1)$ as follows:
$$\psi(x) = A_n e^{ik(x-na)} + B_n e^{-ik(x-na)}.$$
- You can consider solutions that satisfy the following relation (explain, why?)
$$\psi(x+a) = \mu\psi(x).$$
- Write matching conditions at $x = na$ and find a relation between A_n and B_n.
- Prove that $|\mu| = 1$ and use this fact to determine constraints on the energy E.
- Represent μ as $\mu = e^{iqa}$ (q is quasi-momentum). Explain where q is defined.
- Find an implicit equation, which determines the spectrum $E_n(q)$ and analyze it graphically and/or numerically to determine the structure of the bands (n is the band index enumerating energy bands).