The K-homology class of the equivariant Euler characteristic operator

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Abstract

Suppose M^n is a smooth manifold (without boundary) with a smooth action of a finite group G. The de Rham operator $D = d + d^*$ (with respect to a complete G-invariant Riemannian metric), with the grading of forms by parity of degree, gives rise by Kasparov theory to a class $[D] \in KO_0^G(M)$, which when M is closed maps to the (analytical) equivariant Euler characteristic $\chi^G(M)$ in $KO_0^G(\mathrm{pt}) = RO(G)$, the real representation ring. This note, a sequel to an earlier paper on the non-equivariant case, shows that, at least modulo torsion, the information encoded by the class $[D]$ only involves (non-equivariant) Euler characteristics of connected components of fixed sets M^H for cyclic subgroups H of G. In fact $[D]$ is supported on a discrete subset of M, so that it does *not* encode any "higher" equivariant characteristic class data. We show in particular that if M is compact, it is not necessarily true that $[D] = 0$ in $KO_0^G(M)$ if $\chi^G(M) = 0$ in $RO(G)$, but that this *is* true if the fixed set M^H is connected and non-empty for every subgroup H of G. Thus in this case, $[D] \in KO_0^G(M)$ encodes no more information than $\chi^G(M)$.

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1 Introduction and background

Suppose M is a closed C^∞ manifold. In Kasparov's approach [6], an elliptic operator D on M defines a K-homology class $[D]$ on M. The image of this class in K-homology of a point is the index of D. ("Exotic" indices in $KO_1(\mathrm{pt}) = \mathbb{Z}/2$ and equivariant indices in $K_0^G(\mathrm{pt}) = R(G)$, *etc.*, are included in this framework. For a quick discussion of this point of view, see [10]. For excellent more detailed expositions, see [3] and [5].)

The line of research that led to this paper begins with the following:

Motivating Question *What extra topological information (beyond the index) is contained in this K-homology class when D is one of the "standard" geometric operators?*

Rationally, at least in the non-equivariant case, the question is easy to answer. The Atiyah-Singer index formula for D with coefficients in an auxiliary bundle E is of the form

$$\mathrm{Ind}\, D_E = \langle [D], [E] \rangle = \langle \mathcal{I}_D \cup \mathrm{Ch}(E), [M] \rangle,$$

where the first pairing on the right is between K-homology and K-cohomology, the second pairing on the right is between ordinary homology and cohomology, where $\mathcal{I}_D$ is some "index class," a characteristic class associated to D, and where Ch denotes the Chern character. It immediately follows that $\mathrm{Ch}([D]) = [M] \cap \mathcal{I}_D$ is the Poincaré dual of $\mathcal{I}_D$.

Examples

1. D the Euler characteristic operator, $\mathcal{I}_D$ the Euler class of the tangent bundle.
2. D the Dirac operator on a spin manifold, $\mathcal{I}_D$ the $\widehat{A}$-class.
3. D the signature operator on an oriented manifold, $\mathcal{I}_D$ the Atiyah-Singer L-class. (This is basically the Hirzebruch L-class except for some powers of 2.)
4. D the Dolbeault operator on a complex manifold, $\mathcal{I}_D$ the Todd class.

Thus the K-homology class of the signature operator or the Dirac operator encodes all of the rational Pontrjagin classes, and the K-homology class of the Dolbeault operator encodes all of the rational Chern classes. The Euler characteristic operator is different, however, as its K-homology class rationally encodes only the Euler class. In [11], we showed that this is still the case when one takes torsion into account; the K-homology class of the Euler characteristic operator encodes the Euler characteristic and nothing more.

That result suggested looking at the *equivariant* Euler characteristic operator of a G-manifold, when G is a finite group. The information contained in the equivariant K-homology class of this operator is the subject of this paper.

Before proceeding, it is necessary to mention a few additional well known facts. First of all, Kasparov theory defines a K-homology class from an elliptic operator even on a non-compact manifold M, provided that the operator is essentially self-adjoint. This is automatic for the standard operators above if M is equipped with a complete Riemannian metric (or complete Kähler metric, in the case of the Dolbeault operator). Furthermore, for the standard operators D, while D certainly depends on the metric, its K-homology class is independent of the metric chosen, basically because the K-homology class of a "Kasparov bimodule" only depends on the homotopy class of the module, and a homotopy of metrics gives rise to a homotopy of operators. (See the introduction to [11] for more details and references.) However, in the non-compact case, the K-homology class lives in locally finite homology, *i.e.*, the relative Steenrod K-homology of the pair $(\overline{M}, \{\infty\})$, where $\overline{M}$ is the one-point compactification of M and ∞ is the point at infinity. This is the K-homology theory we will be using for non-compact spaces except at one place in this paper (in section 2.3) where we will need the usual homotopy theorist's K-homology for compactly generated spaces which are not necessarily locally compact, which we will denote by K_*^h. (Thus $K_*^h(X) = \varinjlim K_*(Y)$, as Y runs over the compact subsets of X.)

Secondly, when a compact Lie group G acts on M, preserving the chosen Riemannian metric and preserving the operator D, then D defines a class in equivariant K-homology. For the same reasons as in the non-equivariant case, this class is independent of the choice of a complete G-invariant metric. The Kasparov calculus [6] works exactly the same in the equivariant case as in the non-equivariant case (provided G is compact — there is also an equivariant Kasparov theory for G non-compact, but then some major modifications are required). We will restrict attention here to the case that G is finite, not because anything goes seriously wrong for G compact Lie, but simply because

1. this is the most interesting case, since a connected group acts trivially on homology, and
2. the references [17] and [18] on invariant vector fields are only formulated in this context.

From now on D will denote the Euler characteristic operator on a complete Riemannian manifold, or in other words $D = d + d^*$, where d is exterior differentiation and d^* is its adjoint with respect to the Riemannian metric. We view D as an unbounded self-adjoint operator on the $\mathbb{Z}/2$-graded (real) Hilbert space $\mathcal{H} = L^2(\bigwedge T^*M)$ of L^2 sections of the total exterior algebra bundle of the cotangent bundle of M. The grading on $\mathcal{H}$ is the most obvious one — a homogeneous differential form is even if its degree is even, and odd if its degree is odd. Note that D anticommutes with the grading operator.

This paper is basically self-contained, and unlike [11] will not involve any analytical estimates, provided that we make use not just of the main result of [11], but rather of its method of proof. We can paraphrase this as follows:

Theorem 1.1 ([11]) *Let M be manifold without boundary, admitting an everywhere non-vanishing vector field X. Then we can choose a complete Riemannian metric on M and rescale X so that $\|X\| = 1$ everywhere. Once this is done, X can be used to provide a homotopy of self-adjoint operators on $\mathcal{H} = L^2(\bigwedge T^*M)$, anticommuting with the grading operator, from D to an operator commuting with multiplication by functions. This homotopy of operators gives a homotopy of Kasparov modules, from the module defined by D to a degenerate module, and thus $[D] = 0$ in $KO_0(M)$.*

The key thing for us will be the following observation:

Proposition 1.2 *In Theorem 1.1, if a finite group G acts on M, and if the Riemannian metric on M and the vector field X are G-invariant, then the homotopy is also G-invariant, and hence $[D] = 0$ in $KO_0^G(M)$.*

Proof This is immediate since we gave an explicit formula for the homotopy in terms of the vector field X. □

The rest of the paper will be devoted to exploring consequences of Proposition 1.2.

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2 Main Results

2.1 Vanishing Theorems

Using Proposition 1.2, we can now state and prove our main technical result.

Theorem 2.1 *Let G be a finite group and let M be a smooth G-manifold (without boundary) such that for every subgroup H of G, every connected component of M^H is either compact with vanishing Euler characteristic or non-compact. (If some component F of a fixed set M^H is compact, we also have to assume F is not disconnected (at least after dividing out by $N_G(H)$) by any proper fixed set; this is automatic if no fixed set component has codimension 1 in any other.) Then if D is the Euler characteristic operator on M (with respect to a G-invariant complete Riemannian metric), the class $[D]$ of D vanishes in $KO_0^G(M)$.*

Proof This follows from Proposition 1.2 if M admits an everywhere non-vanishing G-invariant vector field. We need to apply the obstruction theory for existence of such vector fields, as developed in [17] and [18]. Waner and Wu show that M always admits a G-invariant vector field X with isolated singularities which they call "canonically transverse," where if x is such a singularity and H is its isotropy group, $X|M^H$ is locally (near x) a sum of a field outward normal to M^H and a field tangent to M^H with index ± 1 at x. Such singularities can be moved around under suitable homotopies, and can also cancel in pairs when two singularities of opposite index coalesce. When M is compact, the obstruction to existence of an everywhere non-vanishing G-invariant vector field is computed in [18], and shown to be given by a certain unstable homotopy-theoretic equivariant Euler characteristic, which we will denote here $\chi_u^G(M)$ to distinguish it from $\chi^G(M)$, the equivariant index of D in $RO(G)$. Waner and Wu also discuss the relationship of $\chi_u^G(M)$ with the stable homotopy-theoretic equivariant Euler characteristic $\chi_s^G(M)$ (discussed, say, in [16], Chapter IV, §2), which takes values in $A(G)$ (the Burnside ring of virtual finite G-sets). Both $\chi_u^G(M)$ and $\chi_s^G(M)$ may be computed from the virtual finite G-set given by the singularities of a G-invariant canonically transverse vector field, where the signs are given by the the indices at the singularities. Incidentally, $\chi^G(M)$ is simply the image of $\chi_s^G(M)$ under the canonical homomorphism $A(G) \to RO(G)$ sending the class of a finite G-set G/H to the class of the associated induced representation $\mathrm{Ind}_H^G 1$.

We now use the hypothesis on fixed sets and proceed by induction over orbit types to get rid of the all the singularities of a G-invariant canonically transverse vector field. Consider the singularities in a component F of a fixed set M^H. We may assume by induction that we have already eliminated singularities in $F \cap M^K$ for $K \supsetneq H$. When F is non-compact, we simply slide the singularities "off to infinity." If F is compact, then by inductive assumption all singularities in F have stabilizer precisely H. Furthermore, we are assuming in this case that

$\chi(F) = 0$, so there must be exactly as many singularities in F with positive index as with negative index. Since the set of points in F with stabilizer precisely H is connected, at least after dividing out by $N_G(H)$, we may cancel the singularities in F as described in [18]. This takes care of the inductive step and completes the proof. $\square$

Corollary 2.2 *Let G be a finite group and let M be a closed G-manifold. Then if the unstable equivariant Euler characteristic $\chi_u^G(M)$ vanishes, so does $[D] \in KO_0^G(M)$.*

Proof The condition implies by [18] that M admits a non-vanishing G-invariant vector field. So Theorem 1.2 applies. $\square$

In the case of free action, things are even easier:

Corollary 2.3 *Let G be a finite group and let M be a connected smooth manifold (without boundary) on which G acts freely. If M is non-compact, or if M is closed and $\chi(M) = 0$, then the class $[D]$ of the equivariant Euler characteristic operator on M vanishes in $KO_0^G(M)$.*

Proof To construct an everywhere non-vanishing G-invariant vector field on M, it suffices to construct a non-vanishing vector field on M/G and to pull it back to the covering space M. Since G is finite, if M is non-compact, so is M/G. And if M is compact, $\chi(M) = |G| \cdot \chi(M/G)$, so $\chi(M) = 0 \iff \chi(M/G) = 0$. Note, incidentally, that this is also precisely the condition for vanishing of $\chi^G(M)$ in this case, since in the case of free action on M, $\chi^G(M) \in RO(G)$ is just $\chi(M/G)$ times the regular representation of G. (One can see this by looking at the finer invariant $\chi_s^G(M) \in A(G)$; recall that $\chi_s^G(M) = \chi_s^G(X)$ for some other finite G-complex X whenever $\chi(M^H) = \chi(X^H)$ for all subgroups H of G ([16], p. 240). But

$$\chi(M^H) = \begin{cases} \chi(M), & H = \{1\}, \\ 0, & \text{otherwise}. \end{cases}$$

This is exactly $\chi(M/G)$ times what we get for $\chi(X^H)$ when X is the finite G-set G, so $\chi_s^G(M) = \chi(M/G) \cdot [G]$ in $A(G)$. Applying the canonical homomorphism $A(G) \to RO(G)$, we see $\chi^G(M)$ is $\chi(M/G)$ times the regular representation of G.) $\square$

In general, we have the following important consequence of Theorem 2.1.

Corollary 2.4 *Let G be a finite group and let M be a smooth G-manifold (without boundary). Then the class $[D]$ of the equivariant Euler characteristic operator on M lies in the image of $\iota_*\colon KO_0^G(F) \to KO_0^G(M)$, where $\iota\colon F \hookrightarrow M$ is the inclusion of some discrete G-invariant subset of M.*

Proof Choose F so that it meets every non-empty compact component of every fixed set M^H, and consider the exact sequence

$$KO_0^G(F) \xrightarrow{\iota_*} KO_0^G(M) \xrightarrow{\text{res}} KO_0^G(M \setminus F).$$

Now $\text{res}([D])$ is just the class of the equivariant Euler characteristic operator on $M \setminus F$ (this requires rescaling the metric—see the comments in [11]), which vanishes by Theorem 2.1, since every fixed set component in $M \setminus F$ is non-compact. Hence $[D]$ lies in the image of ι_*. □

Corollary 2.5 *Let G be a finite group and let M be a closed smooth G-manifold, with the property that $M^G \neq \emptyset$ and that every fixed set M^H, H a subgroup of G, is connected. Then the class $[D]$ of the equivariant Euler characteristic operator on M vanishes in $KO_0^G(M)$ if and only of $\chi^G(M) = 0$ in $RO(G)$.*

Proof Let $x_0 \in M^G$. By assumption, x_0 meets every component of every fixed set M^H, so by the proof of Corollary 2.4, $[D] \in \iota_*(KO_0^G(\{x_0\}))$, $\iota\colon \{x_0\} \hookrightarrow M$ the inclusion. However, ι splits the map $c\colon M \to \text{pt}$, and $\chi^G(M) = c_*([D])$. So $[D] = \iota_*(\chi^G(M))$. □

Corollary 2.6 *Let G be a finite group and let M be a closed smooth G-manifold, with the property that $M^G \neq \emptyset$ and that every fixed set M^H, H a subgroup of G, is connected. Then knowledge of the class $[D] \in KO_0^G(M)$ of the equivariant Euler characteristic operator on M is equivalent to knowledge of $\chi(M^H)$ for each cyclic subgroup H of G.*

Proof This is just a restatement of Corollary 2.5, given the result in [2], p. 574 (see also [16], Lemma IV(2.6)), that if $\chi^G(M)$ is viewed as a virtual character of G, then its value at any $g \in G$ is $\chi(M^g) = \chi(M^{\langle g \rangle})$, where $\langle g \rangle$ is the cyclic subgroup generated by g. □

Remark It is interesting to note that knowledge of the stable homotopy equivariant Euler characteristic, $\chi_s^G(M)$, is equivalent by [18] and [16] to the knowledge of $\chi(M^H)$ for *all* subgroups H of G. But in Corollary 2.6, the Euler characteristic $\chi(M^H)$ of a non-cyclic subgroup H does not appear, even though it is part of the information contained in $\chi_s^G(M)$. Thus there is a sense in which $[D] \in KO_0^G(M)$ is a weaker invariant than $\chi_s^G(M)$ (though we will see in the next subsection that there are cases where it is a stronger invariant). To go further in this direction, we bring restriction into play.

Proposition 2.7 *Let G be a finite group and let M be a closed smooth G-manifold, with the property that M^H is non-empty and connected for every cyclic subgroup H of G. Then if $\chi^G(M) = 0$, the class $[D] \in KO_0^G(M)$ of the equivariant Euler characteristic operator is a ($\mathbb{Z}$-) torsion class.*

Proof We note first of all that since every subgroup of a cyclic group is cyclic, the hypotheses imply that if H is a cyclic subgroup of G, then Corollary 2.5 applies to M viewed by restriction as an H-manifold. Hence the restriction of $[D]$ vanishes in $KO_0^H(M)$ for all H. However, for X a finite G-complex, any class in $K_G^0(X)$ which restricts to 0 in $K_H^0(X)$ for all cyclic $H \subseteq G$ is necessarily ($\mathbb{Z}$-) torsion [13]. To apply that here, simply note that by Poincaré duality, $K_*^G(M)$ may be identified with $K_G^*(T^*M)$, and thus with reduced K_G^* of the Thom space X of the cotangent bundle, and similarly for any subgroup H. So the class $[D] \in K_0^G(M)$ is torsion by Rubinsztein's theorem. The same is true in real K-homology as well, since the kernel of the complexification map from KO_*^G to K_*^G is torsion. $\square$

From Proposition 2.7, we can immediately get an example of a closed G-manifold where $\chi_s^G(M) \neq 0$ (in $A(G)$, which is a free abelian group if we ignore the ring structure), but where the K-homology class of the equivariant Euler characteristic operator is $\mathbb{Z}$-torsion.

Example Let G be the Klein 4-group, the Sylow 2-subgroup of A_4, acting on $V = \mathbb{R}^4$ by permutation of coordinates. (Thus, for example, the permutation $(12)(34)$ acts by $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_4, x_3)$.) Note that for each 2-element cyclic subgroup H of G, $\dim V^H = 2$, while $\dim V^G = 1$. Thus if we let M be the unit sphere in V with the induced G-action, $M^{\{1\}} \cong S^3$, $M^H \cong S^1$ for each 2-element cyclic subgroup H of G, and $M^G \cong S^0$. So $\chi(M^H) = 0$ for cyclic subgroups H of G, while $\chi(M^G) = 2$. Thus $\chi_s^G(M) \neq 0$ in $A(G)$, but $\chi^G(M) = 0$ in $RO(G)$. It follows from Proposition 2.7 that $[D_M] \in KO_0^G(M)$ is torsion, even though $\chi_s^G(M)$ is of infinite order.

The example can be simplified a bit by replacing V by $V \times \mathbb{R}^2$, with G acting trivially on the $\mathbb{R}^2$ factor. This results in having $M^{\{1\}} \cong S^5$, $M^H \cong S^3$ for each 2-element cyclic subgroup H of G, and $M^G \cong S^2$. Again, $\chi_s^G(M) \neq 0$ in $A(G)$, but $\chi^G(M) = 0$ in $RO(G)$. Since all fixed sets in this example are non-empty and connected, $[D] = 0$ in $KO_0^G(M)$ by Corollary 2.5.

2.2 Non-vanishing Theorems

So far it would seem that everything we have done is consistent with the conjecture that when M is connected, $[D] \in KO_0^G(M)$ contains no more information than $\chi^G(M) \in RO(G)$. However, we will see that this is not the case; at least some information about components of fixed sets is encoded in $[D]$ as well.

To prove this, we make use of localization with respect to prime ideals of the representation ring. While this theory (first worked out by Segal [15]) is best known for complex K-theory, it also works with KO^G in place of K^G. I would like to thank J. P. May for providing some references for this fact, which we can summarize as follows:

Theorem 2.8 *Let G be a finite group.*

(1) *If $\mathfrak{p}$ is a prime ideal of $RO(G)$, then there is a unique conjugacy class (H) of cyclic subgroups of G on which $\mathfrak{p}$ is supported. In other words, $\mathfrak{p}$ is the pull-back under restriction $RO(G) \to RO(H)$ of a prime ideal of $RO(H)$ for some cyclic subgroup H of G, and if $\mathfrak{p}$ is also the pull-back of a prime ideal of $RO(K)$ for some other subgroup K of G, then K contains some conjugate of H.*

(2) *Let X be a G-manifold. If X is non-compact, assume X is the interior of a compact G-manifold. (This assumption is stronger than one really needs, but some finiteness assumption is needed in the non-compact case, as pointed out in [12], Remark following Theorem 2.4.) Let $\mathfrak{p}$ and H be as in (1), and let $X^{(H)}$ denote the union of the fixed sets for subgroups conjugate to H,* i.e., *$X^{(H)} = G \cdot X^H$. Then the inclusion $X^{(H)} \hookrightarrow X$ induces an isomorphism on KO_*^G localized at $\mathfrak{p}$.*

Proof If KO is replaced by complex K-theory, then (1) is in [14], §3, and (2) is dual to Proposition 4.1 of [15], and may be found (in slightly greater generality) in [12], Theorem 2.4. The use of the Universal Coefficient Theorem in the proof of the latter may be bypassed (even in the real case) using Poincaré duality as in the proof of Proposition 2.7, for if X is the interior of a compact G-manifold $\overline{X}$, then $KO_*^G(X) \cong KO_*^G(\overline{X}, \partial X) \cong KO_G^*(T^*\overline{X})$, which can be written as reduced KO_G^* of the Thom space of the cotangent bundle of $T^*\overline{X}$.

To complete the argument, we just need the analogues of the results of [14] and [15] for the real case. These may both be found in [1]. For (1), see Remark 3.6 in [1], and for (2), use Theorem 1.2 of [1] in the special case where $I = 0$ and the multiplicative set S is the complement of $\mathfrak{p}$. $\square$

Theorem 2.9 *Let G be a finite group and M a closed G-manifold. Let $\mathfrak{p}$ be a prime ideal of $RO(G)$ and suppose $\mathfrak{p}$ is supported (in the sense of Theorem 2.8) on the conjugacy class of the cyclic subgroup H. Then $[D]_{\mathfrak{p}}$ in $KO_0^G(M)_{\mathfrak{p}}$, KO-homology localized at $\mathfrak{p}$, is a sum of terms coming from the components F_i of $M^{(H)}/G = M^H/N$, where $N = N_G(H)$ is the normalizer of H in G. The contribution from F_i only depends on the germ of $G \cdot \overline{F}_i$ in M as a G-space. (Here if F_i is a component of M^H/N, $\overline{F}_i$ is its preimage in M^H, which might be disconnected.)*

Proof Apply Theorem 2.8. Thus the inclusion

$$M^{(H)} = G \cdot M^H \hookrightarrow M$$

induces an isomorphism on equivariant KO-homology localized at $\mathfrak{p}$. Let U_i be a small G-invariant neighborhood of $G \cdot \overline{F}_i$, small enough so that the various U_i's do not intersect, and let $U = \coprod U_i$. Then $(M \setminus U)^H = \emptyset$, so the restriction map $KO_0^G(M)_{\mathfrak{p}} \to KO_0^G(U)_{\mathfrak{p}} \cong \bigoplus KO_0^G(U_i)_{\mathfrak{p}}$ is an isomorphism. Hence $[D]_{\mathfrak{p}}$ can be computed from the equivariant Euler characteristic operator classes of the open submanifolds U_i. Now one of these classes comes from the Kasparov module defined by D and $\mathcal{H}$, but with the algebra of allowable multiplication operators restricted to continuous functions on M which vanish off U_i. Since such functions do not see what happens off U_i, we could just have well replaced $\mathcal{H}$ by the corresponding Hilbert space for a slightly larger open G-invariant subset V_i of M, and D by the Euler characteristic operator for this open manifold V_i, without changing its Kasparov class. Since U_i and V_i can be chosen as close to $G \cdot \overline{F}_i$ as we like, the result follows. ◻

Using this we can now show that there are cases where $[D]$ encodes more than just χ^G.

Theorem 2.10 *For any non-trivial finite cyclic group G, there exists a closed G-manifold M such that $\chi^G(M) = 0$ (and $\chi_s^G(M) = 0$), but such that the class of the equivariant Euler characteristic operator of M is non-zero in $KO_0^G(M)$ (even in $K_0^G(M)$).*

Proof We will construct a closed 4-dimensional G-manifold M with $\chi(M) = 0$ and such that for any subgroup $H \supsetneqq \{1\}$ of G, $M^H = M^G$ has two components of dimension 2, one of which is S^2 and the other of which is a surface N^2 of genus 2, so that $\chi(N) = -2$. Then

$$\chi(M^H) = \chi(S^2 \amalg N^2) = \chi(S^2) + \chi(N^2) = 2 - 2 = 0,$$

for all subgroups $H \supsetneqq \{1\}$, while also $\chi(M^{\{1\}}) = \chi(M) = 0$, so that $\chi_s^G(M) = 0$ in $A(G)$ and hence $\chi^G(M) = 0$.

For the construction, simply choose any bordism W^3 between S^2 and N^2, and let

$$M^4 = \left(S^2 \times D^2\right) \cup_{S^2 \times S^1} \left(W^3 \times S^1\right) \cup_{N^2 \times S^1} \left(N^2 \times D^2\right).$$

We give this the G-action which is trivial on the S^2, W^3, and N^2 factors, and which is rotation by $\frac{2\pi}{|G|}$ on the D^2 and S^1 factors. Then the action of G is free except for M^G, which consists of $S^2 \times \{0\}$ and of $N^2 \times \{0\}$. Furthermore, we have

$$\begin{aligned} \chi(M) &= \chi\left(S^2 \times D^2\right) - \chi\left(S^2 \times S^1\right) + \chi\left(W^3 \times S^1\right) \\ &\quad - \chi\left(S^2 \times S^1\right) + \chi\left(N^2 \times D^2\right) \\ &= 2 - 0 + 0 - 0 - 2 = 2 - 2 = 0. \end{aligned}$$

Now for any prime ideal $\mathfrak{p}$ of $RO(G)$ not supported at the identity of G, the image $[D]_{\mathfrak{p}}$ of $[D]$ in the $RO(G)$-module $KO_0^G(M_{\mathfrak{p}})$ only depends on the germ of M^G in M, and is a sum of two terms, one coming from an invariant neighborhood of S^2, and the other coming from an invariant neighborhood of N^2. Note that the germ of S^2 in M as a G-manifold is the same as in the G-manifold $M_0^4 = S^2 \times S^2$, with G again acting trivially on the first factor and by rotation around the axis through the two poles on the second factor, and similarly the germ of N^2 in M is the same as in the G-manifold $M_1^4 = N^2 \times S^2$, with G acting trivially on the first factor and by rotation on the second factor. But M_0^G is a disjoint union of two copies of S^2, with G-diffeomorphic neighborhoods, and M_1^G is a disjoint union of two copies of N^2, again with G-diffeomorphic neighborhoods. (Orientation is irrelevant here!) Since G acts trivially on the homology of M, M_0, and M_1, the equivariant Euler characteristics are in all cases the same as the non-equivariant Euler characteristics, with trivial G-action. So a small G-invariant neighborhood of the fixed-set component S^2 must contribute a 2 to $\chi^G(M_0) = 4$ and a small G-invariant neighborhood of the fixed-set component N^2 must contribute a -2 to $\chi^G(M_1) = -4$. That shows that the two pieces of $[D]_{\mathfrak{p}}$ are both non-zero, but cancel each other out in $\chi^G(M)$. $\square$

In fact we can even say more — the equivariant normal data of $M^{(H)}$ is irrelevant, at least as long as $M^{(H)}$ is a manifold. To prove this we start with the "linear case."

Lemma 2.11 *Let G be a finite group, $\mathfrak{p}$ a prime ideal of $RO(G)$ supported on the conjugacy class of a cyclic subgroup H. Let W be a finite-dimensional (real orthogonal) representation of G, such that $W^H = \{0\}$, and let $[D] \in KO_0^G(W)$ be the class of the equivariant Euler characteristic operator on W. Then $[D]_{\mathfrak{p}}$ may be identified with the identity of $RO(G)_{\mathfrak{p}}$.*

Proof By Theorem 2.8, the inclusion of $\{0\} = W^H = W^{(H)}$ into W induces an isomorphism on localized KO-homology, so we may identify $KO_0^G(W)_{\mathfrak{p}}$ with $KO^G(\text{point})_{\mathfrak{p}} = RO(G)_{\mathfrak{p}}$.

Let S be the sphere obtained by gluing together two copies of the unit ball in W along the unit sphere $S(W)$ via the antipodal map. As this intertwines the G-actions, S comes with an action of G for which $S^H = S^0$, the two-element set. Apply Theorem 2.9 to S. Since the germ in S of each of the two H-fixed points is given by W, $[D_S]_{\mathfrak{p}} \in KO_0^G(S)_{\mathfrak{p}} = KO_0^G(S^0)_{\mathfrak{p}}$ is given by two copies of $[D_W]_{\mathfrak{p}}$. On the other hand, it is easy to compute $\chi^G(S)$, since G acts trivially on $H_0(S)$ and either acts trivially or by the sign representation on $H_n(S)$, $n = \dim W = \dim S$ (depending on whether or not G preserves orientation). Now the action of H on W must decompose as a direct sum of irreducible representations, all of dimension 2 or 1. The 2-dimensional irreducible representations of H are by rotation, and preserve orientation, and H has at most two 1-dimensional representations, the trivial representation 1 (excluded here since $W^H = \{0\}$) and the sign representation sgn: $H \twoheadrightarrow \{\pm 1\}$ (which exists only if $|H|$ is even). A direct sum of k copies of the sign representation is orientation-preserving if and only if k is even. So we see that either n is even and H preserves orientation, or else n is odd and H reverses orientation. So when n is even, $\chi^H(S) = 2 \cdot 1$; when n is odd, $\chi^H(S) = 1 - \text{sgn}$, which when viewed as a character on H and evaluated at a generator gives $1 - (-1) = 2$. So in either case, $\chi^G(S)$ takes the value 2 on a generator of H. But $\chi^G(S)$ is obtained by mapping S to a point. The induced map on KO-homology localized at $\mathfrak{p}$ is evidently just the map

$$KO_0^G(S)_{\mathfrak{p}} \cong RO(G)_{\mathfrak{p}} \oplus RO(G)_{\mathfrak{p}} \to RO(G)_{\mathfrak{p}}\colon (x, y) \mapsto x + y.$$

So $2 \cdot [D_W]_{\mathfrak{p}} = 2$ in $RO(G)_{\mathfrak{p}}$, which is torsion-free as an abelian group (since it contains a localization of $\mathbb{Z}$), so $[D_W]_{\mathfrak{p}} = 1$. □

Theorem 2.12 *Let G, M, $\mathfrak{p}$, H, and N be as in Theorem 2.9. Then for any component F_i of M^H/N with preimage $\overline{F_i}$ in M^H for which $G \cdot \overline{F_i}$ is a manifold (this is the case if $N = G$, or if all points in $\overline{F_i}$ have stabilizer precisely H), we can identify the contribution of $G \cdot \overline{F_i}$ to $[D]_{\mathfrak{p}} \in KO_0^G(M)_{\mathfrak{p}}$ with the* *class* $[D_{\overline{F}_i}]_{\mathfrak{q}} \in KO_0^J(\overline{F}_i)_{\mathfrak{q}}$ *of the* J*-equivariant Euler characteristic operator of* $\overline{F}_i$*, where* $J = \{g \in G : g \cdot \overline{F}_i = \overline{F}_i\} \supseteq N$*. (Here* $\mathfrak{q}$ *is a prime ideal of* $RO(J)$*, also supported on* H*, mapping to* $\mathfrak{p}$ *under* $\operatorname{Spec} RO(J) \to \operatorname{Spec} RO(G)$ — see [14], §3 and [1].)

Proof By Theorem 2.9, we can make a preliminary reduction by replacing M by a small G-invariant open neighborhood V of $G \cdot \overline{F}_i$. And clearly V is the disjoint union of $|G/J|$ copies of a small J-invariant open neighborhood U of $\overline{F}_i$, permuted by the action of G. Thus we have a natural isomorphism $KO_*^G(V) \cong KO_*^J(U)$, whereby $KO_*^G(V)_{\mathfrak{p}} \cong KO_*^J(U)_{\mathfrak{q}}$. It remains to show that the inclusion $\overline{F}_i \hookrightarrow U$ (which induces an isomorphism on $\mathfrak{q}$-localized KO-homology by the localization theorem) sends $[D_{\overline{F}_i}]_{\mathfrak{q}}$ to $[D_U]_{\mathfrak{q}}$.

For this we observe that we may identify U (J-equivariantly) with the total space of the J-equivariant normal bundle ν of $\overline{F}_i$. First suppose this normal bundle is trivial, so that we have an J-equivariant splitting $U \cong \overline{F}_i \times W$, where W is a finite-dimensional orthogonal representation of J, with $W^H = \{0\}$. (We will eventually see that the general case is not essentially different.) Since we are free to choose any convenient metric, give W its usual Euclidean metric, and give U the product metric. Then we have an isometric decomposition

$$L^2(\bigwedge T^*U) \cong L^2(\bigwedge T^*(\overline{F}_i \times W)) \cong L^2(\bigwedge T^*(\overline{F}_i)) \otimes L^2(\bigwedge T^*(W)),$$

as well as a corresponding decomposition of d and d^*, so it is clear that $[D_U]$ is simply the external Kasparov product $[D_{\overline{F}_i}]\widehat{\otimes}[D_W]$. But since $W^H = \{0\}$, from Theorem 2.8 we have $RO(J)_{\mathfrak{q}} = KO_0^J(\text{point})_{\mathfrak{q}} \cong KO_0^J(W)_{\mathfrak{q}}$ via the inclusion of the origin into W. Under this isomorphism, $1_{RO(J)}$ corresponds to $[D_W]$, by Lemma 2.11. Multiplication by $1_{RO(J)}$ does nothing, so the inclusion of $\overline{F}_i$ into U sends $[D_{\overline{F}_i}]_{\mathfrak{q}}$ to $[D_U]_{\mathfrak{q}}$.

Even if the normal bundle ν is not trivial, the inclusion of $\overline{F}_i$ into U still induces an isomorphism on KO_0^J localized at $\mathfrak{q}$, by Theorem 2.8, and the local calculation is still as before. So to finish the proof in this case, we redo the argument of Lemma 2.11 by forming the sphere bundle S over $\overline{F}_i$ obtained by gluing together two copies of the unit disk bundle of ν along the copies of the unit sphere bundle by means of the antipodal map. Then S is a J-manifold with $S^H \cong \overline{F}_i \amalg \overline{F}_i$ and with a neighborhood of each copy of $\overline{F}_i$ J-diffeomorphic to U. The bundle projection $S \to \overline{F}_i$ induces a map on equivariant KO-homology which by our previous calculations sends the class of the Euler characteristic operator to the Euler characteristic operator of the base multiplied by the "equivariant Euler characteristic along the fibers," which in turn is calculated in Lemma 2.11. The rest of the argument and the conclusion is the same as when the bundle ν is trivial. $\square$

From this we can now see exactly what information is encoded in $[D] \in KO_0^G(M)$ when G is cyclic.

Theorem 2.13 *Let G be a finite cyclic group and let M be a closed G-manifold. Then the class $[D]$ of the equivariant Euler characteristic operator on M in $KO_0^G(M)$ is $\mathbb{Z}$-torsion (of order dividing a power of $|G|$) if and only if the non-equivariant Euler characteristic vanishes for every connected component of every fixed set M^H, $H \subseteq G$.*

Proof First suppose $\chi(F) \neq 0$ for some component F of some fixed set M^H. Let $\mathfrak{p}$ be a prime ideal of $RO(G)$ supported on H, living over a prime ideal (p) of $\mathbb{Z}$. (The case $p = 0$ is allowed.) By Theorems 2.9 and 2.12, $[D]_{\mathfrak{p}} \in KO_0^G(M)_{\mathfrak{p}}$ is a sum of terms coming from the components of M^H/G. One of these terms is $[D_{\overline{F}}]_{\mathfrak{p}} \in KO_0^G(\overline{F})_{\mathfrak{p}}$, where $\overline{F} = G \cdot F$ is a finite disjoint union of copies of F. Under the map collapsing $\overline{F}$ to a point, $[D_{\overline{F}}]$ maps to $\chi^G(\overline{F})_{\mathfrak{p}} \in RO(G)_{\mathfrak{p}}$. The augmentation map $RO(G) \overset{\text{aug}}{\to} \mathbb{Z}$ induces a map $RO(G)_{\mathfrak{p}} \to \mathbb{Z}_{(p)}$, and the composite

$$KO_0^G(\overline{F})_{\mathfrak{p}} \to RO(G)_{\mathfrak{p}} \overset{\text{aug}}{\to} \mathbb{Z}_{(p)}$$

sends $[D_{\overline{F}}]_{\mathfrak{p}}$ to $\chi(\overline{F})$, some positive multiple of $\chi(F) \neq 0$, just as in the proof of Theorem 2.10. So $[D] \neq 0$.

To prove the converse, we need to show that if $\chi(F) = 0$ for every component F of every fixed set M^H, then $[D]$ is torsion (of order dividing a power of $|G|$) in $KO_0^G(M)$. We prove this by double induction on the dimension of M and on the order of G. We may as well assume M connected. If $\dim M = 0$, there is nothing to prove since a non-empty 0-manifold can never have vanishing Euler characteristic. Similarly, if $G = \{1\}$, the theorem is just a restatement of the result of [11]. So assume by induction that the result holds for manifolds of smaller dimension or for manifolds of the same dimension with smaller cyclic G. We argue by contradiction, assuming $[D] \neq 0$ in $KO_0^G(M) \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ for some prime $p \nmid |G|$. That means $[D]_{\mathfrak{p}} \neq 0$ for some maximal ideal $\mathfrak{p}$ of $RO(G)$ living over $(p) \subset \mathbb{Z}$. (See for example [4], Ch. II, §3.3, Corollary 2 to Theorem 1.) Suppose $\mathfrak{p}$ has support H. That means, by Theorem 2.12, that $[D_{\overline{F}}]_{\mathfrak{p}} \neq 0$ in $KO_0^G(\overline{F})_{\mathfrak{p}}$, with $\overline{F} = G \cdot F$, F some component of M^H. If $\overline{F} \neq M$, *i.e.*, $\dim F < \dim M$, then by inductive assumption, some component of $\overline{F}^K$ has non-zero Euler characteristic for some subgroup K of G. This component is a component of the fixed set for $H \cdot K$, another subgroup of G, contradicting the hypothesis that the Euler characteristic vanishes for every component of every fixed set. The only remaining possibility is that $\overline{F} = M$, which means H acts trivially on M. If $H = \{1\}$, then $\mathfrak{p}$ is the inverse image, under the augmentation map $RO(G) \stackrel{\text{aug}}{\to} \mathbb{Z}$, of the prime ideal (p) of $\mathbb{Z}$. Since $p \nmid |G|$, the character of the regular representation of G (which takes the value $|G|$ at the identity and vanishes elsewhere on G) lies in $RO(G) \setminus \mathfrak{p}$ and annihilates the augmentation ideal. So in this case (by an application of [4], Ch. II, §2.5, Proposition 11), localizing an $RO(G)$-module at $\mathfrak{p}$ means simply tensoring via aug with $\mathbb{Z}_{(p)}$. Thus $[D]_{\mathfrak{p}} \neq 0$ means in particular that the (non-equivariant) class of $[D]$ in $KO_0(M)$ is non-zero. Since $\chi(M) = 0$ by assumption, this contradicts the result of [11]. Thus $H \neq \{1\}$ and our G-action comes from an action of the smaller group $G' = G/H$, so we are again done by inductive hypothesis. □

Theorem 2.13 enables us to strengthen Proposition 2.7.

Proposition 2.14 *Let G be a finite group and let M be a closed smooth G-manifold, with the property that $\chi(F) = 0$ for every connected component F of every fixed set M^H, H a cyclic subgroup of G. Then the class $[D] \in KO_0^G(M)$ of the equivariant Euler characteristic operator is a ($\mathbb{Z}$-) torsion class (of order dividing a power of $|G|$).*

Proof As in the proof of Proposition 2.7, Rubinsztein's theorem [13] shows that to prove $[D]$ is torsion, it is enough to prove $[D] \in KO_0^H(M)$ is torsion for each cyclic subgroup H of G. Furthermore, the proof in [13] shows that a class in KO_G^* which vanishes in KO_H^* for all cyclic H must have exponent dividing $|G|$. This reduces us to the case where G is cyclic, covered by Theorem 2.13. □

2.3 A "Higher G-Index"

As pointed out to me by John Roe, a cute way of reformulating the conclusion of [11] is that "there are no higher Euler characteristics." As we've seen in Theorem 2.10, however, this is false in the equivariant case; it is possible for $\chi^G(M)$ and $\chi_s^G(M)$ to vanish and for the K-homology class of the equivariant Euler characteristic operator to be non-zero. But the counterexample came from disconnectedness of fixed sets. To put things another way, in the equivariant world, π_0 *cannot be disregarded.* However, we will see that this is all there is to the "higher equivariant Euler characteristic." In other words, once π_0 has been accounted for (we do *not* need π_1), the G-index will capture all of the K-homology class of the equivariant Euler characteristic operator, at least modulo torsion. We now proceed to make this precise.

Definition 2.15 If G is a finite group, a G-space X (taken to be G-homotopy equivalent to a G-CW-complex) is called an *equivariant* $K(\pi, 0)$ *space* if, for all subgroups H of G, each non-empty component of X^H is contractible.

An equivariant $K(\pi, 0)$ is in some sense the equivariant analogue of a point, but taking into account the possibility of empty or disconnected fixed sets. Equivariant $K(\pi, 0)$'s show up in many contexts in equivariant homotopy theory; for example, those for which every fixed set is connected (though possibly empty) show up as generators of the idempotents in the Burnside ring ([16], p. 269). An equivariant $K(\pi, 0)$ is also a special case of an "equivariant $K(\pi, 1)$ space," also known as an equivariantly aspherical space, as defined for example in [9] and in [8]; we merely take the fundamental group of each non-empty component of each fixed set to be trivial.

Example If one just looks at semifree G-actions (actions for which the stabilizer of a point is either G or the trivial subgroup) on connected spaces X for which X^G has at most two components, there are (up to equivariant homotopy equivalence) exactly three connected equivariant $K(\pi, 0)$ spaces: a point (the case where X^G is connected and non-empty), EG (the case where $X^G = \emptyset$), and the (unpointed) suspension $SEG = EG \times [0, 1]/\sim$, where $\sim$ collapses $EG \times \{0\}$ and $EG \times \{1\}$ to distinct G-fixed points (the case where X^G has two components).

Theorem 2.16 *Let G be a finite group and let M be a closed G-manifold. Then there is a G-map $f\colon M \to X$ to an equivariant $K(\pi, 0)$ space, such that f induces a bijection of components on all fixed sets. Furthermore, X and f are unique up to equivariant homotopy equivalence, in the sense that given another such map $f'\colon M \to X'$, there is an equivariant homotopy equivalence $h\colon X \to X'$ such that f' is equivariantly homotopic to $h \circ f$.*

Proof This is immediate from the results of [9] and [8], which deal with the harder case of equivariant $K(\pi, 1)$'s. There appear to be some difficulties with Lück's method, but only when G is non-discrete, so the methods of both [9] and [8] work here. $\square$

Definition 2.17 If G is a finite group and M is a closed G-manifold, we define the *improved analytic equivariant Euler characteristic* of M to be $\chi_i^G(M) = f_*([D]) \in KO_0^{G\,h}(X)$, where $f\colon M \to X$ is as in Theorem 2.16. (The notation $KO_*^{G\,h}$ means homotopy-theoretic equivariant K-homology, or $\varinjlim KO_*^G(Y)$, as Y runs over the finite G-subcomplexes of X, as X need not be locally compact and is only defined up to equivariant homotopy equivalence anyway.) Note that if $M^G \neq \emptyset$ and if M^H is connected for all subgroups H of G, then we may take X to be a point and $\chi_i^G(M)$ is the usual analytic equivariant Euler characteristic, $\chi^G(M)$.

Now we may reformulate our main results as follows:

Theorem 2.18 *If G is a finite group and M is a closed G-manifold, then up to ($\mathbb{Z}$-)-torsion, the K-homology class $[D] \in KO_0^G(M)$ of the equivariant Euler characteristic operator carries no more information than $\chi_i^G(M)$. In particular, if $\chi_i^G(M) = 0$, then $[D]$ is a torsion class in $KO_0^G(M)$ (of order dividing a power of $|G|$).*

Proof Choose $f\colon M \to X$ as in Theorem 2.16. For H a subgroup of G, f restricts to a bijection between components of M^H and the (contractible) components of X^H. By Proposition 2.14, it suffices to show that it $\chi_i^G(M) = 0$, then $\chi(F)$ vanishes for every component F of M^H, for all cyclic subgroups H of G. So fix such a cyclic subgroup H and replace G by H. Let $\mathfrak{p}$ be the minimal prime ideal of $RO(H)$ consisting of virtual real representations of H whose characters vanish at all generators of K — see [14], Proposition 3.7 and [1], Remark 3.6. The localization theorem, Theorem 2.8, applies not just to M but to X as well. (This may seem to contradict what was said in [12], Remark following Theorem 2.4, but that was because the theorem was formulated in locally finite homology, whereas in the case of X, we are using $KO_*^{H\,h}(X) = \varinjlim KO_*^H(Y)$, a limit over finite G-subcomplexes.) So if $\chi_i^G(M) = 0$, then $[D]_\mathfrak{p}$ goes to zero in

$$KO_*^{H\,h}(X)_\mathfrak{p} \cong KO_*^{H\,h}(X^H)_\mathfrak{p} \cong \bigoplus_{\text{components of } X^H} RO(H)_\mathfrak{p}.$$

Since

$$[D]_\mathfrak{p} \in KO_0^H(M)_\mathfrak{p} \cong \bigoplus_{\text{components } F \text{ of } M^H} KO_0^H(F)_\mathfrak{p}$$

is by Theorem 2.12 given by the non-equivariant Euler characteristics $\chi(F)$, each of which maps to a different copy of $RO(H)_\mathfrak{p}$, which in turn maps to $\mathbb{Q}$, we can recover each $\chi(F)$ from $\chi_i^H(M)$, and so we are done. ◻

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