BEHAVIOR OF K-THEORY UNDER QUANTIZATION

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1. Introduction

Quantization, at least in some formulations, involves replacing some algebra of observables by a (more non-commutative) deformed algebra. Usually one imposes some additional constraints relating commutation (or anti-commutation) relations in the deformed algebra to a Poisson bracket or related structure on the original algebra. In view of the fundamental role played by K-theory in non-commutative geometry and topology, it is of interest to ask what happens to K-theory, or to its cousin cyclic homology, under this quantization process. A few examples show that in some cases, K-theory ought to behave "rigidly," and that this should have important implications.

Consider first the very simplest example of quantization in a C^*-algebraic context.

1.1 Example. The commutative C^*-algebra $C_0(\mathbb{R}^{2n})$ is isomorphic to the group C^*-algebra of the group $\mathbb{R}^{2n}$, the translational symmetry group of the classical phase space of a single free particle in $\mathbb{R}^n$. In quantum mechanics, the canonical observables $p_1, \dots, p_n$ and $q_1, \dots, q_n$ no longer commute, and instead they satisfy the canonical commutation relations $[p_k, q_j] = i\hbar\delta_{kj}$. The appropriate C^*-algebra replacing $C_0(\mathbb{R}^{2n})$ is thus the quotient of the C^*-algebra of the $(2n+1)$-dimensional Heisenberg group corresponding to the value $\hbar$ of Planck's constant. This is a radically different C^*-algebra; by the Mackey imprimitivity theorem, it is isomorphic to the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space, regardless of the value of $\hbar > 0$. Nevertheless, the C^*-algebras $C_0(\mathbb{R}^{2n})$ and $\mathcal{K}$ have the same K-theory, $\mathbb{Z}$ in even degrees and 0 in odd degrees. That the classical and quantum C^*-algebras have the same K-theory is not a trivial result; it is equivalent to the Bott Periodicity Theorem.

1.2 Example. Alternatively, one can work in a purely algebraic context. From this point of view, the classical system just described corresponds to the polynomial ring $B_n = \mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n]$, while the corresponding quantum mechanical system corresponds to the Weyl algebra A_n, the non-commutative

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polynomial algebra on the same generators $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$, but subject to the canonical commutation relations $[p_k, q_j] = i\hbar\delta_{kj}$. Since we are dealing with untopologized algebras, it no longer makes sense to talk about topological K-theory, but we can compare the *algebraic* K-theory of the two algebras B_n and A_n. As is well known, algebraic K-theory of regular Noetherian rings (in particular, of fields) is unchanged under polynomial extensions, so the K-theory of B_n is just that of the field $\mathbb{C}$. Quite remarkably, the same is true of the K-theory of A_n; this is a consequence of the same machinery that is used to compute the higher K-theory of B_n, namely, Quillen's results on the K-theory of filtered rings ([24, Theorem 7]–see also [17]).

1.3 Example. There is also a fermionic analogue of these examples. We can consider the non-commutative polynomial algebra on generators $p_1, \ldots, p_n$, $p_{n+1} = q_1, \ldots, p_{2n} = q_n$, subject to the canonical *anti*-commutation relations $p_k p_j + p_j p_k = 2\delta_{kj}$. This is of course the complex Clifford algebra on $2n$ generators, which is isomorphic to a matrix algebra, and thus has the same K-theory as the commutative polynomial ring. (The Clifford algebra on an *odd* number of generators, however, splits up as a direct sum of two matrix algebras, and has the same K-theory as $\mathbb{C} \oplus \mathbb{C}$.)

Unfortunately, I do not know of any uniform explanation for the phenomena indicated by these examples. Thus this paper will be primarily a survey of results in a number of different areas, all with a common theme. Section 2 is devoted to the links between results about K-theory of quantum C^*-algebras and the index theory of Toeplitz and pseudodifferential operators. Section 3 concerns results about K-theory in the context of C^*-algebras defined by deformation quantization. Section 4 is devoted to the purely algebraic theory of uncompleted quantum algebras over $\mathbb{C}$, along with some applications to index theory.

2. C^*-algebraic Quantization and Index Theory

Among the most important examples of quantization are those associated with Toeplitz and pseudodifferential operators. The noncommutative multiplication of such operators is to be viewed as a quantum analogue of the commutative multiplication of the associated symbol functions. The simplest non-trivial example of a C^*-algebra arising from such a construction is the "Toeplitz algebra" $\mathcal{T}$, the C^*-algebra generated by the unilateral shift operator. This algebra also arises as the C^*-algebra generated by Toeplitz operators $T_f = PM_f$ with symbols f in $C(S^1)$, S^1 viewed as the unit circle in $\mathbb{C}$, acting on the Hardy space $H^2(S^1)$ (the L^2 closure of the polynomials in z). (Here M_f is multiplication by f on $L^2(S^1)$ and P is orthogonal projection from $L^2(S^1)$ onto $H^2(S^1)$.) The map $f \mapsto T_f$ is linear but not multiplicative, but it is well known that the inverse map $T_f \mapsto f$ extends to a homomorphism $\mathcal{T} \to C(S^1)$ fitting into a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0, \tag{2.1}$$

for the which the connecting map of the associated long exact sequence in K-theory,

$$\partial : \mathbb{Z} \cong K^{-1}(S^1) = K_1(C(S^1)) \to K_0(\mathcal{K}) \cong \mathbb{Z},$$

is an isomorphism. It follows that $K_1(\mathcal{T}) = 0$ and that $K_0(\mathcal{T}) \cong \mathbb{Z}$. The interpretation of the map ∂ is that if $f: S^1 \to \mathbb{C}$ is everywhere non-vanishing (so that it defines a class $[f]$ in $K_1(C(S^1))$ corresponding to the winding number of f around 0), then T_f is a Fredholm operator, and $\partial([f])$ is the negative of its Fredholm index. Hence one recovers the famous result of Gohberg and Krein that $\operatorname{index} T_f = -\text{winding number}(f)$.

If one views $\mathcal{T}$ as being a kind of non-commutative version of $C(S^1)$, then the calculation of the K-theory of $\mathcal{T}$ might be disappointing, since the natural map $\mathcal{T} \to C(S^1)$ is far from inducing an isomorphism on K-theory. However, it has been pointed out by many authors, particularly Berger and Coburn [1, 5] and Klimek and Lesniewski [15, 16], that there are other ways of looking at this example that match better with the philosophy of preservation of K-theory under quantization. From this perspective, we are to view $\mathcal{T}$ as a quantization of an algebra of functions on either the closed unit disk $D = D^2 \subset \mathbb{C}$ or the complex plane $\mathbb{C}$. The quantization is achieved by replacing a function f by the associated Toeplitz operator $T_f = PM_f$, where this time M_f is multiplication by f on $L^2(D, \mu)$ or $L^2(\mathbb{C}, \mu)$, for μ some absolutely continuous measure, and P is orthogonal projection onto the Bergman space (the subspace of holomorphic functions in L^2). The operator T_f acts on the Bergman space $H^2 = PL^2$. Different variants of the construction come from taking μ to be area measure on D, giving the classical Bergman space, or to be area measure weighted by some power of $(1-|z|^2)^{-1}$, which is sometimes better behaved under the action of $SU(1, 1)$, or to be Gaussian measure on $\mathbb{C}$, giving "Berezin quantization" on "Segal-Bargmann space." But in all cases, there are some symbol functions f for which the Toeplitz operator T_f is compact, and there is a larger class of symbol functions f for which the associated operators T_f generate a C^*-algebra isomorphic to $\mathcal{T}$. For example, in the case of the classical Bergman space on D, T_f is compact if f is continuous on D and vanishes on the boundary circle S^1, and the T_f for f continuous on the closed disk generate an algebra isomorphic to $\mathcal{T}$, so that $\mathcal{T}$ may be viewed as a "non-commutative closed disk." Indeed, $\mathcal{T}$ has the same K-theory as D ($\mathbb{Z}$ in even degrees, 0 in odd degrees). In the case where μ is a symmetrical Gaussian measure on $\mathbb{C}$, the results of [1, 2] show that the T_f, for f continuous and bounded on the whole plane and having continuous limits on the circle at infinity (i.e., with $\lim_{r\to\infty} f(r, \theta)$ a well-defined continuous function of θ, where r and θ are polar coordinates), generate a copy of $\mathcal{T}$ for which the map $\mathcal{T} \to C(S^1)$ corresponds to sending T_f to the limiting values of f at infinity. This time $\mathcal{T}$ may be viewed as a non-commutative version of the disk compactification of $\mathbb{C}$, and again it has the same K-theory. This equality of K-theory groups is basically equivalent to a version of the Gohberg-Krein index theorem for Berezin-Toeplitz operators, which is discussed in [2].

There is a huge literature on C^*-algebras generated by generalized Toeplitz operators, and much of it can also be approached from the point of view of "preservation of K-theory under quantization." Just as an example, let us cite [13], which defines a family of Toeplitz C^*-algebras $\mathcal{T}$ associated to irrational flows on $\mathbb{T}^n$, acting on the positive spectral subspace in $L^2(\mathbb{T}^n)$ for the generator of the flow. The algebra $\mathcal{T}$ fits into a short exact sequence

$$0 \to \mathcal{C} \to \mathcal{T} \to C(\mathbb{T}^n) \to 0, \tag{2.2}$$

where this time $\mathcal{C}$ is not type I, and it again turns out that $K_1(\mathcal{T}) = 0$ and that $K_0(\mathcal{T}) \cong \mathbb{Z}$. One should interpret this as saying that $\mathcal{T}$ represents a kind of very non-commutative quantization of the polydisk $(D)^n \subset \mathbb{C}^n$, for which the K-theory is the same as for $(D)^n$. Again, the K-theory calculation is equivalent to a sort of (non-classical) index theorem for the relevant class of Toeplitz operators.

Other examples of quantization, more closely related to Weyl quantization than to Berezin quantization, come from the theory of pseudodifferential operators. The classical set-up, described more carefully in [14], is this. Suppose M^n is a closed manifold with cotangent bundle T^*M. The symbols of classical pseudodifferential operators of order ≤ 0 are certain C^∞ functions on T^*M which have radial limits at infinity in each cotangent space and with appropriate asymptotic expansions. The closure of the space of symbols is the algebra $C(D(T^*M))$, where $D(T^*M)$ is the compactification of T^*M obtained by attaching a copy of S^{n-1} over each point of M. As a topological space, $D(T^*M)$ is homeomorphic to the unit disk bundle of T^*M, and $D(T^*M) \smallsetminus (T^*M)$ is homeomorphic to the unit sphere bundle $S(T^*M)$ of T^*M. The closure of the symbols of pseudodifferential operators of order ≤ -1 is the subalgebra $C_0(T^*M)$ of functions vanishing at infinity in each cotangent space. Since pseudodifferential operators of order ≤ 0 act by bounded operators on $L^2(M)$ (defined using the smooth measure coming from a Riemannian metric), we can take their closure in the operator norm, getting a C^*-algebra $\mathcal{P}(M)$ acting on $L^2(M)$. Furthermore, since pseudodifferential operators of order ≤ -1 act by compact operators, we obtain short exact sequences of C^*-algebras:

$$0 \to C_0(T^*M) \to C(D(T^*M)) \to C(S(T^*M)) \to 0 \tag{2.3}$$

and

$$0 \to \mathcal{K} \to \mathcal{P}(M) \xrightarrow{\sigma} C(S(T^*M)) \to 0, \tag{2.4}$$

where the second sequence is the "quantization" of the first one and σ is the map sending an operator to its principal symbol, a function on the sphere bundle. This time, the algebras $\mathcal{P}(M)$ and $C(D(T^*M))$ do not have the same K-theory, but there is an interesting relationship between the long exact sequences in K-theory coming from the two short exact sequences (2.3) and (2.4). The connecting maps of the two long exact sequences fit into a commuting diagram

$$\begin{array}{ccccccc} K_1(C(S(T^*M))) & \xrightarrow{\partial} & K_0(C_0(T^*M)) & \xrightarrow{\cong} & K_0(M) & & \\ \| & & \big\downarrow{\scriptstyle \mathrm{ind}_t} & & \big\downarrow & & \\ K_1(C(S(T^*M))) & \xrightarrow{\partial} & K_0(\mathcal{K}) & \xrightarrow{\cong} & K_0(\mathrm{pt}) & \xrightarrow{\cong} & \mathbb{Z} \end{array} \tag{2.5}$$

which has the following interpretation. An operator $P \in M_k(\mathcal{P}(M))$ is Fredholm (on $L^2(M)^k$) if and only if its principal symbol $\sigma(P)$ is everywhere invertible on $S(T^*M)$, i.e., if and only if P is elliptic. When this is the case, $\sigma(P)$ defines a class $[\sigma(P)]$ in $K_1(C(S(T^*M))) = K^{-1}(S(T^*M))$ and P defines a class $[P] \in K_0(M)$. Furthermore, $\partial([\sigma(P)]) \in K_0(\mathcal{K}) \cong \mathbb{Z}$ in the second sequence is by definition the analytic index of P, and $\partial([\sigma(P)]) \in K_0(C_0(T^*M)) \cong K^0(T^*M)$ (K-theory with compact supports) in the first sequence is the K-theoretic class of the symbol. The map $K_0(C_0(T^*M)) \xrightarrow{\cong} K_0(M)$ in the first sequence is homotopy-theoretic Poincaré duality,[1] and the downward map $K_0(M) \to K_0(\mathrm{pt})$ is induced by collapsing M to a point. Commutativity of the right-hand square is true by definition of the topological index map ind_t, and commutativity of the left-hand square amounts to the Atiyah-Singer Index Theorem, in other words, to the assertion that the analytic index of P is equal to the topological index.

There are many more examples of C^*-algebras that can be viewed as quantizations coming from some sort of pseudodifferential calculus, and the calculation of the K-theory and its relation to the K-theory of the algebra of symbols can again often be interpreted in terms of an index theorem. A good survey of many of these situations may be found in [27].

3. C^*-algebraic Deformation Quantization

It is often useful to consider not just a single quantization of a classical system but rather a one-parameter family of quantizations, indexed by the values of Planck's constant, so that the *correspondence principle* holds: the classical system is recovered in the limit as $\hbar \to 0$. Such one-parameter families of quantizations, when they satisfy certain other desirable axioms, are often called *deformation quantizations*. The purely algebraic theory of deformation quantization will be treated in Section 4, but here we focus attention on the C^*-algebraic version of this theory, which has been made completely rigorous by Rieffel [25, 27].

3.1 Definition (Rieffel [25]). Suppose A is a C^*-algebra with a dense $*$-subalgebra $\mathcal{A}$, and one is given a Poisson bracket $\{\ ,\ \}$ on $\mathcal{A}$, satisfying $\{a, b\}^* = \{b^*, a^*\}$. Then a *strict deformation quantization* of $\mathcal{A}$ is given by an interval $I = [0, \hbar_0]$ and by families of products $\star_\hbar$, involutions $^{*_\hbar}$, and C^*-algebra norms $\|\ \|_\hbar$ (compatible with the involutions $^{*_\hbar}$) on $\mathcal{A}$, with $\hbar \in I$, which for $\hbar = 0$ give the original C^*-algebra A, and which satisfy:

1. For each $a \in \mathcal{A}$, $a \mapsto \|a\|_\hbar$ and $a \mapsto a^{*_\hbar}$ are continuous on I (this has the consequence that the C^*-algebra completions of $\mathcal{A}$ for the various values of $\hbar \in I$ give a continuous field of C^*-algebras over I);
2. For each $a, b \in \mathcal{A}$, $\|\frac{1}{\hbar}(a \star_\hbar b - ab) - \{a, b\}\|_\hbar \to 0$ as $\hbar \to 0$.

In many cases of such deformation quantizations, the completions of $\mathcal{A}$ at all non-zero values of $\hbar$ turn out to be isomorphic to one another, at least for $\hbar$ sufficiently small. In such cases, one can compare Rieffel's definition with a more abstract notion of deformation.

3.2 Definition (Connes-Higson [6]). Let A and B be C^*-algebras. A *deformation* of A into B is a continuous field of C^*-algebras over some closed

[1] If M has a spin^c structure—for example, if M is a stably almost complex manifold—then there is a Thom isomorphism $K^i(M) \xrightarrow{\cong} K^{n+i}(T^*M)$, and Poincaré duality takes the more familiar form of an isomorphism $K^{n-i}(M) \to K_i(M)$ given by cap-product with a fundamental class.

interval $I = [0, \hbar_0]$, whose fiber over 0 is A, and whose fibers $A_\hbar$ for $\hbar > 0$ make up the constant field over $(0, \hbar_0]$ with fiber B. Note that the algebra of continuous sections of such a deformation is a C^*-algebra extension

$$0 \to C_0((0, \hbar_0], B) \to C \to A \to 0.$$

Connes and Higson point out that every deformation from A into B gives rise to a morphism $K_*(A) \to K_*(B)$, and that when A is K-nuclear (this is usually the case in applications; in fact, A is often commutative), a deformation from A into B gives a class in Kasparov's group $KK(A, B)$. Conversely, Connes and Higson show that every "asymptotic morphism" from A to B (a norm-continuous family of maps which "in the limit" are linear, $*$-preserving, and multiplicative) comes from a deformation from A/J into B, for some closed two-sided ideal $J \triangleleft A$.

Many of the examples which we discussed in Section 2 can be put into one or the other of these frameworks, often both. We can say that a deformation in the sense of Connes-Higson is *K-theoretically rigid* if the induced morphism $K_*(A) \to K_*(B)$ is an isomorphism. (When A and B are in the category for which the Universal Coefficient Theorem for E-theory holds in the sense of [30], this implies that the associated element in $E(A, B)$ has an inverse in $E(B, A)$.) Then to give a few illustrations, there is a K-theoretically rigid deformation of $A = C(D)$ into the Toeplitz algebra $B = \mathcal{T}$ corresponding to the quantization discussed in Section 2. This may be constructed by letting $C_\hbar$ be the C^*-algebra generated by an element z with $\|z\| \leq 1$ satisfying $zz^* = (1 - \hbar)z^*z + \hbar 1$ (the fibers for $0 < \hbar < 2$ are all isomorphic to $\mathcal{T}$—see for example [20]). On the other hand, the Weyl pseudodifferential operator quantization gives a deformation of $C_0(T^*M)$ into $\mathcal{K}$ which corresponds to the index map but is not K-theoretically rigid. (This example is cited in [6].)

Now we can state the main result of [7], which is based on [18].

3.3 Theorem (Dădărlat-Loring [7]**).** *If X is a closed orientable manifold with one point removed, and if B is a σ-unital C^*-algebra for which there is an isomorphism (simply of $\mathbb{Z}/2$-graded abelian groups) $K_*(C_0(X)) \to K_*(B)$, then this isomorphism is induced by a K-theoretically rigid deformation from $C_0(X)$ into $B \otimes \mathcal{K}$.*

Thus K-theoretically rigid deformations are quite common. For example, this theorem would guarantee the existence of a K-theoretically rigid deformation from $C_0(\mathbb{R}^{2n})$ to $\mathcal{K}$ corresponding to the Bott element (cf. Example 1.1), though the construction would not be very explicit. Most deformations, however, do not correspond to quantizations and are therefore not physically interesting. Problems which are not so easy to answer are the following:

(1) Given a quantization with some physical interest, when can it be formulated as a strict deformation quantization in the sense of Definition 3.1? as a deformation in the sense of Definition 3.2?
(2) When a deformation "arises from physics," is there some reasonable (physical) criterion which will force it to be K-theoretically rigid?

The work of Rieffel gives some fragmentary positive answers to these questions, in [25] to question 1 and in [26] to question 2.

4. The Purely Algebraic Theory

In some cases, especially in quantum field theory, C^*-algebras may not provide the correct framework for quantization, or the estimates to set up the theory in a C^*-algebraic context may not be available. This is not so surprising in view of the fact that the Poisson bracket for functions on a symplectic manifold is (at least initially) only defined for smooth functions. Furthermore, while quantization as we know it seems to require that observable quantities correspond to operators on a Hilbert space, in many cases the most basic quantities (including position and momentum) correspond to *unbounded* operators. While it is possible that an optimal theory of quantization should involve Fréchet algebras, and that K-theoretic rigidity should be studied in the context of topological K-theory of such algebras, we will for simplicity restrict attention here to purely algebraic versions of quantization, in which the topology of the algebras of observables is ignored. Note that if one is only interested in K_0, this makes no difference in the results, since *algebraic* K_0 satisfies excision and agrees with topological K_0.

Let us begin with a generalization of Example 1.2 of the Weyl algebra. The following is known to algebraic K-theorists (see [3] and [17]), though perhaps its physical interpretation is not as familiar as it should be.

4.1 Theorem [24]. *Let V be a smooth affine algebraic variety over $\mathbb{C}$, and let $\mathcal{D}(V)$ be the ring of algebraic linear differential operators on V. (If we view V as sitting in affine space, then these are simply differential operators with rational coefficients holomorphic on V.) Then the algebraic K-theory of $\mathcal{D}(V)$ is canonically isomorphic to that of $\mathcal{O}(T^*V)$, the ring of algebraic functions on the (algebraic) cotangent bundle.*

The Weyl algebra arises as the special case of $\mathcal{D}(V)$ when V is (complexified) configuration space $\mathbb{C}^n$, for which T^*V is (complexified) phase space. In general, one may view $\mathcal{D}(V)$ as being a quantization of $\mathcal{O}(T^*V)$, which is the algebra of "symbols" for differential operators. The theorem thus says that this example of quantization is K-theoretically rigid in a suitable sense. Thus, at least after passage to K-theory, one has a correspondence between algebraic vector bundles over phase space T^*V and "quantum vector bundles," that is, finitely generated projective modules over $\mathcal{D}(V)$.

The proof of Theorem 4.1 is actually more general than the result, so that one has analogous theorems, which we will not bother to state, for other examples of algebraic quantization of physical interest, such as the passage from $U(\mathfrak{g})$ (for $\mathfrak{g}$ a complex simple Lie algebra) to its quantum analogue, $U_q(\mathfrak{g})$. In addition, these results have analogues for cyclic homology, which say that cyclic homology also remains unchanged under passage to quantization ([3], [12], [32]).

As an alternative to studying just a single quantization of a classical system (corresponding to fixing a single value of $\hbar$), it is even more natural to study a whole one-parameter family of quantizations all at once, or in other words, to study deformation quantization. In the literature, this has mostly only been done in a formal sense; that is, in the realm of formal power series expansions. Thus we have:

4.2 Definition. Let $A_0 = (\overline{A}_0, \cdot)$ be an algebra over $\mathbb{C}$, where $\overline{A}_0$ is the underlying vector space of A_0 and $\cdot$ is the multiplication in A_0. A (formal) *deformation quantization* of A_0 will mean an (associative) algebra $A = (\overline{A}_0[[\hbar]], \star)$ over $\mathbb{C}[[\hbar]]$ (the commutative ring of formal power series) with underlying $\mathbb{C}[[\hbar]]$-module $\overline{A}_0[[\hbar]]$, where the multiplication $\star$ in A is defined by perturbing the multiplication $\cdot$ in A_0 to a new product $\star$ via

$$a \star b = a \cdot b + \hbar\phi_1(a, b) + \hbar^2\phi_2(a, b) + \cdots, \qquad a, b \in \overline{A}_0,$$

and then extending to series in the obvious way:

$$\left(\sum_{j=0}^{\infty} a_j \hbar^j\right) \star \left(\sum_{l=0}^{\infty} b_l \hbar^l\right) = \sum_{j,\, l=0}^{\infty} \hbar^{j+l} \left(a_j \cdot b_l + \sum_{p=1}^{\infty} \hbar^p \phi_p(a_j, b_l)\right).$$

Here ϕ_j, $j = 1, 2, \ldots$ are $\mathbb{C}$-bilinear maps $\overline{A}_0 \times \overline{A}_0 \to \overline{A}_0$. Note that $A_0 \cong A/(\hbar)$ *as algebras*, so that one has a natural algebra map $e_0 : A \twoheadrightarrow A_0$ ("setting $\hbar$ to 0"). We call the map e_0 the *classical limit map*.

4.3 Example. A trivial but still important example is the case where the multiplication in A_0 is undeformed, i.e., $\star = \cdot$. In this case $A = A_0[[\hbar]]$ is simply a ring of formal power series in one variable over the ring A_0.

4.4 Example. In one of the most important examples, $A_0 = C^\infty(M)$, where M is a symplectic manifold. Then there exist non-commutative algebras A satisfying Definition 4.2 for which

$$\phi_1(f, g) - \phi_1(g, f) = \{f, g\},$$

where $\{\ ,\ \}$ is the Poisson bracket on M. This was shown in [8] and [10]. The $\star$-product is obtained by a patching procedure using the Weyl quantization of $C^\infty(\mathbb{R}^{2n})$.

4.5 Example. There is a very interesting subcase of Example 4.4 in which M is a "quantizable" compact Kähler manifold. (Here "quantizable" means there is a holomorphic line bundle L on M with a connection and metric compatible with ω; in particular, $c_1(L) = [\omega]$ and some power of L is very ample.) In this case, an explicit deformation quantization of $A_0 = C^\infty(M)$ (not depending on any "patching" arguments) is constructed in [31] using Berezin-Toeplitz operators on the spaces of holomorphic sections of powers $L^{\otimes m}$ of L. Since these spaces of holomorphic sections are finite-dimensional, the algebra generated by Toeplitz operators on any one of them is just a finite-dimensional matrix algebra, but the classical limit corresponds to letting $m \to \infty$.

In the generality of Definition 4.2, it turns out that K_0 is preserved under deformation quantization. Higher K-theory is not (this is clear from looking at K_1 in the context of Example 4.3—the local ring $\mathbb{C}[[\hbar]]$ has far more units than $\mathbb{C}$ itself, so $(e_0)_* : K_1(\mathbb{C}[[\hbar]]) \to K_1(\mathbb{C})$ is far from being injective—its kernel is the multiplicative group of power series with constant term 1), but we can recover rigidity if we use K-theory with finite coefficients, which washes out "divisible junk."

4.6 Theorem [28]**.** *Let A_0 be a $\mathbb{C}$-algebra (with unit), let A be a deformation quantization of A_0 in the sense of Definition 4.2, and let $e_0 : A \twoheadrightarrow A_0$ be the associated classical limit map. Then the maps $(e_0)_* : K_0(A) \to K_0(A_0)$ and $(e_0)_* : K_j(A; \mathbb{Z}/m) \to K_j(A_0; \mathbb{Z}/m)$, induced by e_0, are isomorphisms (for all $m > 1$, $j > 0$).*

This theorem was motivated by the work of Nest and Tsygan [21, 22] on "formal" index theorems, which depends heavily on an analogue of Theorem 4.6 for cyclic homology. For a quick summary of this work, see [29].

We conclude this section by discussing how to get from this result to a proof of the Atiyah-Singer index theorem and some related results. Other similar arguments may be found in [23], [11], and [9]. Let M^n be a compact smooth manifold without boundary, and let $X = T^*M$, its cotangent bundle. We have certain subspaces of $C^\infty(X)$ which are symbol classes for (classical) pseudodifferential operators on M. In particular, let $A_0 = \mathcal{S}^0(M)$ and let $B_0 = \mathcal{S}^{-\infty}(M)$, the spaces of symbols for pseudodifferential operators on M of orders 0 and $-\infty$, respectively. Recall that B_0 is just the classical Schwartz space of X, that is the space of smooth functions which, together with all their derivatives, vanish rapidly at infinity (with respect to the vector space structure on cotangent spaces), while A_0 consists of functions f for which (in local coordinates (x, ξ), where x is a local coordinate on M and ξ is a local coordinate on a cotangent space) $\partial_x^\alpha \partial_\xi^\beta f = O(|\xi|^{-|\beta|})$ for all multi-indices α and β. These two symbol spaces are both algebras under pointwise multiplication, and B_0 is an ideal in A_0, so we can form the quotient $C_0 = A_0/B_0$. Now define the $\star$-product as in [23, Definition 2.4 and 2.5]. This thereby defines deformation quantizations A, B, and C of A_0, B_0, and C_0, respectively, with $B \triangleleft A$ and with $C \cong A/B$. To explain what is going on, let $(\rho_\hbar p)(x, \xi) = p(x, \hbar\xi)$ (in local coordinates), for $p \in A_0$. This defines a "rescaling" on all the symbol spaces. Then for $p, q \in A_0$, $p \star q \in A$ is the asymptotic expansion (in $\hbar$) of $\rho_{\hbar^{-1}}(r_\hbar)$, where $\rho_\hbar p$, $\rho_\hbar q$, and $r_\hbar$ are the "total symbols" of the pseudodifferential operators $P_\hbar$, $Q_\hbar$, and $R_\hbar = P_\hbar Q_\hbar$ (operator product), respectively. (The idea of doing all this comes from [4].)

We are now ready to discuss the index theorem. Let P be a $k \times k$ matrix-valued pseudodifferential operator of order 0 on M with total symbol $p \in M_k(A_0)$, and assume that P is elliptic. This means precisely that p is invertible off a compact neighborhood of the 0-section in $X = T^*M$. Define $\rho_\hbar p$ and the corresponding asymptotic family of operators $\mathcal{P} = \{P_\hbar\}_\hbar$ as above. Since P is elliptic, it acts as a Fredholm operator (say on $C^\infty(M)^k$), and our problem is to compute its Fredholm index in terms of the symbol p. Since we can find a smooth parametrix for P which is also pseudodifferential of order 0, it is easy to see that the family $\mathcal{P}$ is invertible modulo $M_k(B)$, and thus defines a class $[\mathcal{P}] \in K_1(C)$. By the same reasoning as in the analysis of the diagram (2.5), the index of P may be recovered from $\partial([\mathcal{P}]) \in K_0(B)$ by taking the "trace" τ in the sense of [23, Proposition 2.5(3)].

Now let $e_0 : K_0(B) \to K_0(B_0)$ be the classical limit map. By Theorem 4.6, this is an isomorphism. Furthermore, since B_0 is just the Schwartz space of T^*M, we clearly have $K_0(B_0) \cong K^0(T^*M)$ (topological K-theory with compact support). Also, since the classical limit map comes from setting $\hbar \to 0$ in the asymptotic symbol, in other words, taking the *principal* symbol, we have

$$(e_0)_*(\partial([\mathcal{P}])) = [p], \tag{4.7}$$

the K-theory class of the principal symbol. So the analytic index of P agrees with the result of applying to $[p]$ the composite map

$$K^0(T^*M) \cong K_0(B_0) \xrightarrow{((e_0)_*)^{-1}} K_0(B) \xrightarrow{\tau_*} \mathbb{Z}. \tag{4.8}$$

So to prove the index theorem, it suffices to identify (4.8) with the topological index. This is now a formal, though quite non-trivial, calculation—all the analysis has already been taken care of in the previous steps. As pointed out in [23, Theorem 3.1], the details are in [21].

The usual form of the index theorem only involves the part of Theorem 4.6 dealing with K_0. However, the part dealing with K_1 also has an index-theoretic interpretation. In the above context, it implies that the map $e_0 : C \to C_0$ induces an isomorphism on K_1 with finite coefficients. The asymptotic family of elliptic operators $\mathcal{P}$ has a "determinant" in $K_1(C; \mathbb{Z}/m)$ for each m, so we see by Theorem 4.6 that this may be identified with the class of the symbol modulo smoothing symbols in $K_1(C_0; \mathbb{Z}/m)$. Here we are using algebraic, not topological, K-theory, but the methods of [19, §7], for example, show that if one uses finite coefficients there is no difference, so $K_1(C_0; \mathbb{Z}/m) \cong K^{-1}(S(T^*M); \mathbb{Z}/m)$. And a class in $K^{-1}(S(T^*M))$ vanishes if and only if it is zero mod m for all m (since the K-groups of $S(T^*M)$ are all finitely generated). One may thus check that $\mathcal{P}$ is stably (with respect to letting $k \to \infty$) smoothly homotopic to the identity within the class of asymptotic elliptic operators if and only if the class of its principal symbol vanishes in $K^{-1}(S(T^*M))$. It would be quite interesting to find similar concrete applications of the *higher* K-theory of C in degrees ≥ 2.

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