Groupoid C^*-algebras and index theory on manifolds with singularities

Jonathan Rosenberg (jmr@math.umd.edu) *
Department of Mathematics, University of Maryland
College Park, MD 20742-4015, USA

Abstract. The simplest case of a manifold with singularities is a manifold M with boundary, together with an identification $\partial M \cong \beta M \times P$, where P is a fixed manifold. The associated singular space is obtained by collapsing P to a point. When $P = \mathbb{Z}/k$ or S^1, we show how to attach to such a space a noncommutative C^*-algebra that captures the extra structure. We then use this C^*-algebra to give a new proof of the Freed-Melrose $\mathbb{Z}/k$-index theorem and a proof of an index theorem for manifolds with S^1 singularities. Our proofs apply to the real as well as to the complex case. Applications are given to the study of metrics of positive scalar curvature.

Keywords: groupoid C^*-algebra, manifold with singularities, $\mathbb{Z}/k$-manifold, elliptic operator, KK-theory, index theorem, positive scalar curvature.

Mathematics Subject Classification (MSC2000): Primary 58J22. Secondary 19K56, 46L87, 46L85, 46L80.

1. Introduction

The simplest example of a "manifold with singularities" in the sense of Sullivan ([26], cf. also [18]) and Baas [3] is a $\mathbb{Z}/k$-manifold, a manifold with boundary whose boundary consists of k identical components, all identified with one another. These were originally introduced for the purpose of giving a geometric meaning to bordism with $\mathbb{Z}/k$ coefficients, or to index invariants with values in $\mathbb{Z}/k$, such as the "signature mod k" (see [18]). Later, Freed [8] and Freed-Melrose [9] were able to give an analytic version of a index theorem for such manifolds, and other (or, as some might claim, better) proofs were given by Higson [11], Kaminker-Wojciechowski [14], and Zhang [27]. The innovation in Higson's proof was the use of noncommutative C^*-algebras to model the operator-theoretic part of the index calculation.

However, the "philosophy of noncommutative geometry" would suggest still another approach. Namely, a $\mathbb{Z}/k$ structure on a manifold should be modeled by a noncommutative C^*-algebra, and then one should work with this C^*-algebra just as one would work with the usual algebra of functions on a manifold in proving the index theorem.

* Partially supported by NSF grant DMS-0103647.

The purpose of this paper is to implement such an approach, using the concept of a groupoid C^*-algebra as introduced by Renault ([20], or one can also find a nice exposition in [19]). This has several ancillary benefits. First of all, it treats the $\mathbb{Z}/k$ index just as one treats other kinds of indices in Kasparov-style index theory, following the outline that one can find in [21]. Secondly, it suggests a program for extending everything to manifolds with other kinds of singularities, of which a good general exposition can be found in [5]. This has potential geometric applications as explained for example in [6].

This paper arose out of lectures I prepared for the Summer Research Conference on Noncommutative Geometry at Mount Holyoke College in June, 2000. It's a pleasure to thank the organizers of that meeting, Alain Connes, Nigel Higson, and John Roe, as well as the sponsor of the lectures, the Clay Mathematics Institute, for their help and support. Some of the results were also presented at the Workshop on Noncommutative Geometry and Quantization at MSRI in April, 2001. I would also like to thank the referee for a number of useful suggestions.

2. The C^*-algebra of a $\mathbb{Z}/k$-Manifold

DEFINITION 2.1. Let $\Sigma = (P)$, where P is a closed (smooth) manifolds (not necessarily connected). Then a (closed) Σ-*manifold*, or *manifold with singularities* Σ, means a compact manifold M with boundary, together with a diffeomorphism $\partial M \xrightarrow{\cong} \beta M \times P$, for some closed manifold βM. In case M and P are oriented or have a spin structure, this diffeomorphism is required to respect the additional structure. Perhaps the most interesting special case is the one where P is 0-dimensional, i.e., is the disjoint union of k points. In this case, we call M a $\mathbb{Z}/k$-manifold. This case is illustrated in Figure 2.1, with $k = 3$.

The situation we've been discussing can be generalized to the case where $\Sigma = (P_1, \cdots, P_r)$ consists of more than one kind of singularity. For more details, see [3], [5], or [6].

Note that while M itself is smooth, there is a singular space M_Σ associated to it, obtained by collapsing ∂M to βM by collapsing each $\{x\} \times P$, $x \in \beta M$, to a point. Note the map $M \twoheadrightarrow M_\Sigma$ is injective on the interior of M.

Roughly speaking, the C^*-algebra, $C^*(M; \Sigma)$, of a manifold M with singularities Σ should be the algebra of functions on the singular quotient space M_Σ. However, this is too coarse an invariant, as it doesn't take all the extra structure into account. Instead, since M_Σ is the quotient of M by an equivalence relation $\sim$ encoding the Σ-structure

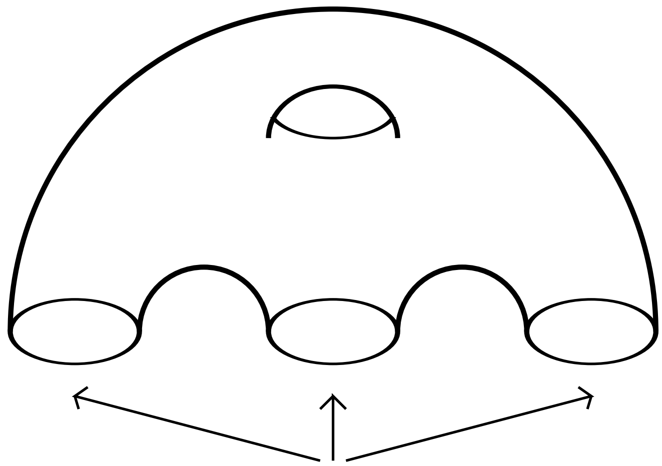

Figure 2.1. A $\mathbb{Z}/3$-manifold

on M, we want to use something like the C^*-algebra of the equivalence relation. Here we are following the philosophy of noncommutative geometry as expounded in [7]: points in the same equivalence class should "talk to each other" but not be collapsed to one. However, this is still not exactly right, for a number of reasons:

1. Later we will want to do analysis without having to worry about the boundary. For this reason (following a trick in [11]) we add half-infinite cylinders onto the boundary first, as in Figure 2.2.

2. We need to construct the C^*-algebra so that it has the "right" K-theory. For instance, in the case of a $\mathbb{Z}/k$-manifold, $K_*(C^*(M;\Sigma))$ should be related to K-theory of M with $\mathbb{Z}/k$ coefficients.

3. We want to construct $C^*(M;\Sigma)$ as the C^*-algebra of a suitable locally compact groupoid, in the sense of [20]. Such a groupoid is not so hard to construct in the case of a $\mathbb{Z}/k$-manifold (see Definition 2.2), but in the case of general singularities, it is not clear how to proceed.

DEFINITION 2.2. Let $k \geq 2$ and let M be a compact $\mathbb{Z}/k$-manifold, that is a manifold with singularities $\Sigma = (\mathbb{Z}/k)$ in the sense of Definition 2.1. Recall that this means that, as part of the given structure on M, we have a diffeomorphism $\phi\colon \partial M \to \beta M \times \mathbb{Z}/k$, for some closed manifold βM. Let $N = M \cup_{\partial M} \partial M \times [0, \infty)$, where $\partial M \subset M$ is identified to

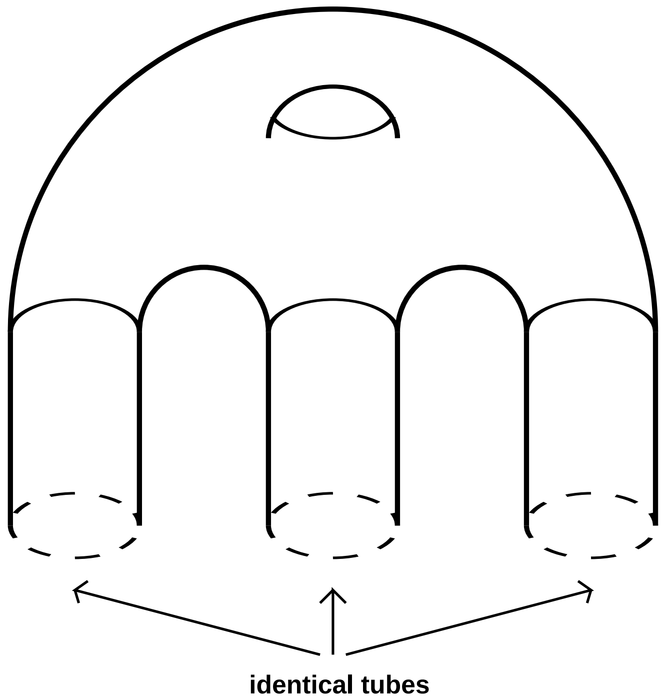

Figure 2.2. A $\mathbb{Z}/3$-manifold with cylinders added

$\partial M \times \{0\}$. This is a manifold without boundary, usually non-compact. Note that N is homeomorphic to the interior of M, i.e., to $M \smallsetminus \partial M$, via the collaring theorem. We define an equivalence relation $\sim$ on N as follows. Points in M (including points in ∂M) are equivalent only to themselves. A point $(x, t) \in \partial M \times (0, \infty)$ is equivalent to a point $(y, s) \in \partial M \times (0, \infty)$ if and only of $t = s$ and $p(x) = p(y)$, where $p\colon \partial M \to \beta M$ is $\phi\colon \partial M \to \beta M \times \mathbb{Z}/k$ followed by projection onto the first factor.

Let $G \subset N \times N$ be the equivalence relation $\sim$ viewed as a groupoid. We observe that G is locally closed in $N \times N$. Indeed, the closure $\overline{G}$ of G in $N \times N$ is easily seen to consist of $G \cup G'$, where G' is the equivalence relation on ∂M identifying points which project to the same point in βM. So

$$\begin{aligned} \overline{G} \smallsetminus G &\cong \{(x, i, j) \mid x \in \beta M,\ 1 \le i,\ j \le k,\ i \ne j\} \\ &\cong \beta M \times ((k^2 - k) \text{ points}) \end{aligned}$$

is compact and thus G is open in $\overline{G}$, so that G is locally compact in the relative topology from $N \times N$. (Note that this argument would break down in the case where $\Sigma = (P)$, $\dim P > 0$.) The unit space of the groupoid G is of course N, and the range map $r\colon G \to N$ is a local homeomorphism, since this is obvious over the interior of M and over $\partial M \times (0,\infty)$, whereas the only points in G over $\partial M \times \{0\}$ are of the form (x,x), $x \in \partial M$, whose small neighborhoods in G have projection under r that miss all but one of the cylinders $\beta M \times \{j\} \times [0,\infty)$. Hence by [20, Proposition I.2.8], G has a Haar system which is essentially unique, and the C^*-algebra $C^*(G)$ is well-defined. We denote it by $C^*(M;\mathbb{Z}/k)$. Note, incidentally, that in the construction of $C^*(M;\mathbb{Z}/k)$, one can use either real or complex scalars. When it is necessary to distinguish the real and complex C^*-algebras, we will denote the former by $C^*_{\mathbb{R}}(M;\mathbb{Z}/k)$ for emphasis.

For future use, we also define two additional C^*-algebras. We let $C^*(\mathbb{Z}/k;\mathbb{Z}/k)$ denote the C^*-algebra of the locally compact principal groupoid G_1 with unit space $(\mathbb{Z}/k) \times [0,\infty)$ defined by the equivalence relation $\sim$ with $(x,t) \sim (y,s)$ if and only if $t = s$ and either $x = y$ or else $t = s > 0$. This is well defined for the same reason as $C^*(M;\mathbb{Z}/k)$, and again we can use either real or complex scalars. Finally, we let $C^*(\mathrm{pt};\mathbb{Z}/k)$ denote the C^*-algebra of the locally compact principal groupoid G_2, the quotient of G_1 obtained by collapsing $(\mathbb{Z}/k) \times \{0\}$ to a single point.

PROPOSITION 2.3. *Let $k \geq 2$ and let M be a compact $\mathbb{Z}/k$-manifold. Then $C^*(M;\mathbb{Z}/k)$ fits into a short exact sequence of C^*-algebras*

$$0 \to C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_k \to C^*(M;\mathbb{Z}/k) \to C(M) \to 0. \tag{2.1}$$

This is valid with either real or complex scalars.

Similarly we have exact sequences

$$0 \to C_0(\mathbb{R}) \otimes M_k \to C^*(\mathbb{Z}/k;\mathbb{Z}/k) \to C(\mathrm{pt})^k \to 0. \tag{2.2}$$

and

$$0 \to C_0(\mathbb{R}) \otimes M_k \to C^*(\mathrm{pt};\mathbb{Z}/k) \to C(\mathrm{pt}) \to 0. \tag{2.3}$$

Proof. Note that M is closed in N and on N the equivalence relation $\sim$ of Definition 2.2 is trivial. So we obtain (see [20, Proposition II.4.5]) a quotient C^*-algebra of $C^*(M;\mathbb{Z}/k)$ isomorphic to $C(M)$, and the kernel of the quotient map $C^*(M;\mathbb{Z}/k) \to C(M)$ must be attached to the complementary open set, $\partial M \times (0,\infty)$. Since the inverse image of this set in G splits as a product $\beta M \times (0,\infty) \times (\mathbb{Z}/k) \times (\mathbb{Z}/k)$, and the C^*-algebra of the groupoid $(\mathbb{Z}/k) \times (\mathbb{Z}/k)$ is just M_k, the first result follows. The cases of $C^*(\mathbb{Z}/k;\mathbb{Z}/k)$ and $C^*(\mathrm{pt};\mathbb{Z}/k)$ are handled exactly the same way. □

PROPOSITION 2.4. *Let $k \geq 2$ and let M be a connected compact $\mathbb{Z}/k$-manifold. Then the connecting map in the long exact K-theory sequence associated to the extension* (2.1) *may be identified with the map $\iota^*\colon K^*(M) \to K^*(\beta M)$, followed by multiplication by k. Here $\iota\colon \beta M \to M$ denotes the inclusion. If complex scalars are used, K^* means KU^*, and if real scalars are used, K^* means KO^*.*

Proof. The connecting map of (2.1) in K-theory is

$$\begin{aligned} K_i(C(M)) &\cong K^{-i}(M) \to \\ &K_{i-1}\big(C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_k\big) \cong K_{i-1}\big(C_0(\mathbb{R}) \otimes C(\beta M)\big) \\ &\cong K^{-i+1}(\mathbb{R} \times \beta M) \cong K^{-i}(\beta M). \end{aligned} \tag{2.4}$$

Now let $C^*(\partial M; \mathbb{Z}/k)$ be defined like $C^*(M; \mathbb{Z}/k)$, in the sense that we use an equivalence relation on $\partial M \times [0, \infty)$ which is trivial on $\partial M \times \{0\}$ and identifies points in $\partial M \times (0, \infty)$ having the same image in $\beta M \times (0, \infty)$. Then in analogy with extension (2.1) we have an extension

$$0 \to C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_k \to C^*(\partial M; \mathbb{Z}/k) \to C(\partial M) \to 0 \tag{2.5}$$

in which $C(\beta M)$ splits off as a tensor factor. In other words, $C^*(\partial M; \mathbb{Z}/k) \cong C(\beta M) \otimes C^*(\mathbb{Z}/k; \mathbb{Z}/k)$. Now because of the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_k \arrow[r] \arrow[d, equal] & C^*(M; \mathbb{Z}/k) \arrow[r] \arrow[d] & C(M) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ 0 \arrow[r] & C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_k \arrow[r] & C^*(\partial M; \mathbb{Z}/k) \arrow[r] & C(\partial M) \arrow[r] & 0 \end{tikzcd}$$

the map of (2.4) factors through the restriction map $K^{-i}(M) \to K^{-i}(\partial M)$, followed by connecting map for (2.5), which in turn is the external product of the identity map on $K^{-i}(\beta M)$ with the connecting map for (2.2). But $K^*(\partial M) \cong K^*(\beta M)^k$, and the map $K^*(M) \to K^*(\partial M)$ is just $\iota^*\colon K^*(M) \to K^*(\beta M)$, followed by the diagonal inclusion of $K^*(\beta M)$ into a product of k copies of itself. So we can rewrite the map of (2.4) as the composite of $\iota^*\colon K^*(M) \to K^*(\beta M)$ with the external product of the identity map on $K^*(\beta M)$ with the connecting map for (2.3), the result of collapsing the k copies of the scalars in (2.2) to one.

We also only need to compute the connecting map in K_0. (That's because the connecting map for (2.3) is easily seen to be a map of $K_*(\text{pt})$-modules from $K_*(\text{pt})$ to itself, and so it's determined by what happens to the generator in degree 0.) Now one computes that

$$C^*(\text{pt}; \mathbb{Z}/k) = \{f \in C_0([0, \infty), M_k) \mid f(0) \text{ a multiple of } I_k\}.$$

This is simply the mapping cone of the inclusion of the scalars into M_k as multiples of the $k \times k$ identity matrix, so the connecting map is multiplication by k, as required. □

REMARK 2.5. In fact Propositions 2.3 and 2.4 are also valid when $k = 1$, but in this case, $\partial M = \beta M$ and $C^*(M; \mathbb{Z}/k)$ is simply $C_0(N)$. Since N is properly homotopy equivalent to the interior of M, its K-theory is just the relative K-theory of the pair $(M, \partial M)$, and so all statements are obvious in this case.

COROLLARY 2.6. *The K-theory of the C^*-algebra $C^*(\mathrm{pt}; \mathbb{Z}/k)$ defined in* (2.3) *is just $K_i(C^*(\mathrm{pt}; \mathbb{Z}/k)) \cong K^{-i-1}(\mathrm{pt}; \mathbb{Z}/k)$. The dual theory* (*often known as "K-homology," though it is a* cohomology *theory on C^*-algebras*) *is given by $K^{-i}(C^*(\mathrm{pt}; \mathbb{Z}/k)) \cong K_i(\mathrm{pt}; \mathbb{Z}/k)$.*

Proof. This is obvious from Proposition 2.4, the long exact sequences, and the universal coefficient theorem. □

REMARK 2.7. One might note that the algebra $C^*(\mathrm{pt}; \mathbb{Z}/k)$ is in some sense (cf. [23]) dual to the algebra $\mathcal{D}_k(\mathcal{H})$ defined in [11]. That algebra (say in the complex case) has K_0-group $\cong \mathbb{Z}/k$ and vanishing K_1; our $C^*(\mathrm{pt}; \mathbb{Z}/k)$ has K^0-group $\cong \mathbb{Z}/k$ and vanishing K^1.

3. The C^*-algebra of an η-Manifold

As mentioned above, we are not sure how to give a good definition of $C^*(M; \Sigma)$ in the case of general singularities. However, there is one interesting case in which we can give an *ad hoc* definition, the case of an η-manifold.

DEFINITION 3.1. An *η-manifold* (cf. [6]) is a manifold with singularities $\Sigma = (\eta)$ in the sense of Definition 2.1, where η denotes S^1 with its non-bounding spin structure. If we forget the spin structure, an η-manifold is just a manifold M with singularities $\Sigma = (S^1)$, but later we will require M to have a spin structure inducing a product spin structure $(\beta M, s) \times \eta$ on its boundary. (Here s is a spin structure on βM.) As is well known, η generates $\Omega_1^{\mathrm{Spin}} \cong \mathbb{Z}/2$, and is the image of the generator of $\pi_1^s \cong \mathbb{Z}/2$ with the same name.

DEFINITION 3.2. Let M^n be an η-manifold in the sense of Definition 3.1. Thus M is a compact manifold with boundary, together with a diffeomorphism $\phi\colon \partial M \to \beta M \times S^1$ (and some spin structure data). Let $N = M \cup_{\partial M} \partial M \times [0, \infty)$, where $\partial M \subset M$ is identified to $\partial M \times \{0\}$.

This is a manifold without boundary, usually non-compact. Let N' be the quotient space of N (locally compact, but not a manifold) obtained by collapsing $\partial M \subset M$ to βM via the projection map $\beta M \times S^1 \to \beta M$. Note that the algebra $C_0^{\mathbb{R}}(N')$ can be identified with the subalgebra of functions $f \in C_0^{\mathbb{R}}(N)$ which are constant along $\{x\} \times S^1$ for each $x \in \beta M$. Identify S^1 with the unit circle in the complex plane and let the group S^1 act on N' as follows: the action is trivial on the interior of M and on the image of ∂M, and S^1 acts on

$$\partial M \times (0, \infty) \cong \beta M \times S^1 \times (0, \infty)$$

by the formula $z \cdot (x, w, t) = (x, zw, t)$, $x \in \beta M$, $z, w \in S^1$, $t > 0$. Alternatively, the image of $\partial M \times [0, \infty)$ in N' can be identified with $\beta M \times \mathbb{C}$, with a product action where S^1 acts trivially on βM and on $\mathbb{C}$ by multiplication. (Here $\beta M \times \{0\}$ corresponds to the image of $\partial M \times \{0\}$, and $\beta M \times \{z \in \mathbb{C} \mid |z| = t\}$ is identified with $\partial M \times \{t\}$ when $t > 0$.) The fixed point set for the S^1-action on N' is the singular space M_Σ (M with $\partial M \cong \beta M \times S^1$ collapsed to βM).

Let $C^*_{\mathbb{R}}(M; \eta)$ and $C^*(M; \eta)$ be the real and complex transformation C^*-algebras $C_0^{\mathbb{R}}(N') \rtimes S^1$ and $C_0(N') \rtimes S^1$. Also let $C^*_{\mathbb{R}}(\mathrm{pt}; \eta)$ denote $C_0^{\mathbb{R}}(\mathbb{C}) \rtimes S^1$, and similarly with $C^*(\mathrm{pt}; \eta)$. We have a canonical S^1-equivariant proper "collapse" map $c \colon N' \to \mathbb{C}$ sending M to 0 and projecting

$$\beta M \times \{z \in \mathbb{C} \mid |z| = t\} \to \{z \in \mathbb{C} \mid |z| = t\}$$

for $t > 0$, so this induces a dual map of C^*-algebras $C^*_{\mathbb{R}}(\mathrm{pt}; \eta) \to C^*_{\mathbb{R}}(M; \eta)$, and similarly in the complex case.

PROPOSITION 3.3. *The S^1-equivariant real K-theory of $\mathbb{C}$ with compact supports (for the action of S^1 by multiplication) may be identified with the K-theory of $C^*_{\mathbb{R}}(\mathrm{pt}; \eta)$, and is given by*

$$KO_i(C^*_{\mathbb{R}}(\mathrm{pt}; \eta)) \cong KO^{-i}_{S^1}(\mathbb{C}) \cong KO^{-i-2}_{S^1}(\mathrm{pt})$$

for all i (though one has periodicity mod 8). These groups are computed in [2]. *The (Kasparov) dual theory is given by*

$$KO^{S^1}_i(\mathbb{C}) \cong KO^{S^1}_{i-2}(\mathrm{pt})$$

for all i (again with periodicity mod 8). Similarly, $K^{S^1}_i(\mathbb{C}) \cong R(S^1) = \mathbb{Z}[t, t^{-1}]$ for i even, 0 for i odd.

Proof. The identification of equivariant K-theory (for a compact group action) with K-theory of the crossed product may be found in [13]. The proof is only stated for the complex case, but everything goes over the real case as well. The rest of this is equivariant Bott periodicity, for which one can see [24] for the case of K-cohomology, and [15], §5, for the case of Kasparov K-homology. □

REMARK 3.4. The reader might wonder what would happen if we used the analogue of the definition of $C^*(M;\eta)$ in the case of a $\mathbb{Z}/k$-manifold. In other words, if M is a $\mathbb{Z}/k$-manifold and $N = M \cup_{\partial M} \partial M \times [0,\infty)$, we can form N', the result of collapsing $\partial M \times \{0\} \cong \beta M \times \mathbb{Z}/k \times \{0\} \subset N$ to $\beta M \times \{0\}$. Note that N' contains M_Σ as a closed subset, and $N' \smallsetminus M_\Sigma \cong \beta M \times \mathbb{Z}/k \times (0,\infty)$. Also $\mathbb{Z}/k$ acts on N' semi-freely, the action being free on $\beta M \times \mathbb{Z}/k \times (0,\infty)$ with quotient space $\beta M \times (0,\infty)$, and trivial on M_Σ. Thus we obtain an exact sequence

$$0 \to C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_k \to C(N') \rtimes \mathbb{Z}/k \to C(M_\Sigma)^k \to 0.$$

Thus the result of this construction would be quite similar to $C^*(M;\mathbb{Z}/k)$ as defined in Definition 2.2. More precisely, there is a commuting diagram with exact rows, built out of two pull-back squares:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_k \arrow[r] \arrow[d, equal] & C^*(M;\mathbb{Z}/k) \arrow[r] & C(M) \arrow[r] & 0 \arrow[d, equal] \\
0 \arrow[r] \arrow[d, equal] & C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_k \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[u] \arrow[d] & C(M_\Sigma) \arrow[r] \arrow[u, hook] \arrow[d, hook, "\Delta"] & 0 \arrow[d, equal] \\
0 \arrow[r] & C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_k \arrow[r] & C(N') \rtimes \mathbb{Z}/k \arrow[r] & C(M_\Sigma)^k \arrow[r] & 0,
\end{tikzcd} \tag{3.1}$$

with $\Delta\colon C(M_\Sigma) \hookrightarrow C(M_\Sigma)^k$ the diagonal inclusion and $C(M_\Sigma) \hookrightarrow C(M)$ induced by $M \twoheadrightarrow M_\Sigma$.

4. Three index theorems

DEFINITION 4.1. Let M be a closed Σ-manifold, where $\Sigma = (P)$, $P = \mathbb{Z}/k$ or η. Recall that M comes with a diffeomorphism $\phi\colon \partial M \to \beta M \times P$. For purposes of this section, an elliptic operator D on M will mean an elliptic pseudodifferential operator which on a small collar neighborhood of the boundary, $\partial M \times [0,\varepsilon)$, is invariant under translation normal to the boundary (i.e., is the restriction of a $\mathbb{R}$-invariant elliptic operator on $\partial M \times \mathbb{R}$) and is also invariant under translation in P. (Recall that in our case, P is a compact group, either $\mathbb{Z}/k$ or S^1, and we can identify $\partial M \times [0, \varepsilon)$ with $\beta M \times P \times [0, \varepsilon)$ via ϕ.) The (graded) vector bundle on which such an operator acts must satisfy a similar invariance condition. The primary examples of such operators are the *standard elliptic operators*: the Euler characteristic operator of a Riemannian manifold, the signature operator of an oriented Riemannian manifold, the Dirac operator of a spin (or spin^c) Riemannian manifold, and the Dolbeault operator of a Kähler manifold. In all cases, the associated Riemannian metric has to be a product metric (coming from a metric on βM and from the standard metrics on P and $\mathbb{R}$) in a neighborhood of the boundary.

DEFINITION 4.2. **(the analytic $\mathbb{Z}/k$-index)** Let $(M, \phi\colon \partial M \xrightarrow{\cong} \beta M \times \mathbb{Z}/k)$ be a closed $\mathbb{Z}/k$-manifold, and let D be an elliptic operator on M in the sense of Definition 4.1. We assume D is acting on sections of some vector bundle, equipped with a grading and suitable Clifford module data so that D locally defines a class in K_i for some i, where K_* denotes either real or complex K-homology, as appropriate. The main examples we have in mind are the following:

- M^i is a spin manifold and $(\beta M)^{i-1}$ has the induced spin structure. D is the $\mathcal{C}\ell_i$-linear real Dirac operator in the sense of [17], Chapter II, §7, and Chapter III, §16. ($\mathcal{C}\ell_i$ denotes the Clifford algebra of $\mathbb{R}^i$.) In this case D locally defines a class in KO_i.

- M^{2n} is a spin^c manifold and $(\beta M)^{2n-1}$ has the induced spin^c structure. D is the complex Dirac operator acting on the spinor bundle with $\mathbb{Z}/2$-grading defined by the half-spinor bundles. In this case D locally defines a class in K_0.

- M^{2n} is an oriented even-dimensional manifold and D is the signature operator as in [17], Chapter II, §6, which locally defines a class in K_0.

- M^{2n} is a complex manifold (of complex dimension n) with a Kähler metric which near the boundary is compatible with the $\mathbb{Z}/k$-structure, and D is the Dolbeault operator. In this case D locally defines a class in K_0.

Since, in a neighborhood of the boundary, D is invariant under translations normal to the boundary of M, it naturally extends to an elliptic operator on the manifold N of Definition 2.2. We claim that D defines a class $[D] \in K^{-i}(C^*(M; \mathbb{Z}/k)) = KK(C^*(M; \mathbb{Z}/k), \mathcal{C}\ell_i)$. Indeed, this is obvious from the fact that D defines a $C_0(N)$-$\mathcal{C}\ell_i$ Kasparov bimodule which is equivariant for the groupoid G (in Definition 2.2), and thus gives a Kasparov bimodule for the groupoid C^*-algebra. Since a homotopy of metrics gives rise to a homotopy of operators, the Kasparov class $[D]$ is independent of the choice of metric. Note that via the open inclusions $\mathbb{R} \times \beta M \hookrightarrow \widehat{C^*(M;\mathbb{Z}/k)}$ and $\operatorname{int} M \hookrightarrow \widehat{C^*(M;\mathbb{Z}/k)}$, $[D]$ restricts to the usual class defined by the relevant elliptic operator on the open manifold $\mathbb{R} \times \beta M$ or $\operatorname{int} M$. The *analytic* $\mathbb{Z}/k$*-index* $\operatorname{ind}_{\mathrm{a}}(D)$ of D is defined to be the image of the class $[D]$ under the map $c_*\colon K^{-i}(C^*(M;\mathbb{Z}/k)) \to K^{-i}(C^*(\mathrm{pt};\mathbb{Z}/k)) \cong K_i(\mathrm{pt};\mathbb{Z}/k)$. The map c_* is dual to the inclusion

$$c^*\colon\ C^*(\mathrm{pt};\mathbb{Z}/k) \hookrightarrow C^*(M;\mathbb{Z}/k), \tag{4.1}$$

or equivalently can be viewed as the map in "K-homology" induced by the "collapse map" c of M to a point. The identification of $K^{-i}(C^*(\mathrm{pt};\mathbb{Z}/k))$ comes from Corollary 2.6.

DEFINITION 4.3. **(the topological $\mathbb{Z}/k$-index)** Let notation be as in Definition 4.2 above. Let $[\sigma(D)] \in K^*(T^*M)$ or $KR^*(T^*M)$, the K-theory with compact supports of the cotangent bundle of M, be the class of the principal symbol of the operator. In the real case we need to view T^*M as a Real space, as explained in [1], or in [17, Chapter III, §16]. Note that $[\sigma(D)]$ is invariant under the identifications on the boundary, i.e., it comes by pullback from the quotient space T^*M_Σ (the image of T^*M with the k copies of $T^*M|_{\beta M}$ collapsed to one) under the collapse map $M \twoheadrightarrow M_\Sigma$. Following [8] we define the *topological* $\mathbb{Z}/k$*-index* $\operatorname{ind}_{\mathrm{t}} D$ of D as follows. Start by choosing an embedding $\iota\colon (M, \partial M) \hookrightarrow (D^{2r}, S^{2r-1})$ of M into a ball of sufficiently large even dimension $2r$ ($2r$ divisible by 8 in the real case), for which ∂M embeds $\mathbb{Z}/k$-equivariantly into the boundary (if we identify S^{2r-1} with the unit sphere in $\mathbb{C}^r$, $\mathbb{Z}/k$ acting as usual by multiplication by roots of unity). First consider the complex case, with D anti-commuting with a $\mathbb{Z}/2$-grading so as to give a class in K_0. Recall that if M were a closed manifold, the usual topological index of D would be the image of $[\sigma(D)]$ under the push-forward map on complex K-theory

$$\iota_!\colon\ K^0(T^*M) \to K^0\big(T^*(D^{2r}, S^{2r-1})\big) \cong K^0(D^{2r}, S^{2r-1}) \cong \mathbb{Z},$$

or in other words the composite

$$K^0(T^*M) \xrightarrow[\cong]{\text{Poincaré duality}} K_0(M) \xrightarrow{\iota_*} K_0(D^{2r}, S^{2r-1}) \cong K^0(D^{2r}, S^{2r-1}).$$

We simply want the analogue of this with the $\mathbb{Z}/k$-structure taken into account, or in other words, the result of a push-forward map

$$\widehat{\iota_!}\colon\ K^0(T^*M_\Sigma) \to K^0\big(T^*(D^{2r}_\Sigma, S^{2r-1}_\Sigma)\big) \cong \widetilde{K}^0(M^{2r}_k) \cong \mathbb{Z}/k,$$

where M_k^{2r} is the Moore space obtained by dividing out by the $\mathbb{Z}/k$-action on the boundary of D^{2r}. We call the image the topological index of D, $\mathrm{ind_t}(D)$. The push-forward is defined by the "Thom map" followed by extension by 0; in other words, we have the Thom isomorphism $K^0(T^*M) \to K_0(T^*\nu)$, where ν is a tubular neighborhood of M in D^{2r}, and since this is equivariant for the $\mathbb{Z}/k$-structure on the boundary, it descends to a map

$$K^0(T^*M_\Sigma) \to K_0(T^*\nu_\Sigma) \to K^0\big(T^*(D^{2r}_\Sigma, S^{2r-1}_\Sigma)\big).$$

It is shown in [9] that this push-forward map is uniquely characterized by natural analogues of the Atiyah-Singer axioms. The real case is similar, except that we need to use the push-forward map on Real K-theory instead, getting a topological index in $KO^*(\mathrm{pt}; \mathbb{Z}/k)$.

THEOREM 4.4 (cf. [8], [9], [11], and [14]). *Let* $(M, \phi\colon \partial M \xrightarrow{\cong} \beta M \times \mathbb{Z}/k)$ *be a closed* $\mathbb{Z}/k$*-manifold, and let* D *be an elliptic operator on* M *in the sense of Definition 4.1. Then the analytic index of* D *in* $K_i(\mathrm{pt}; \mathbb{Z}/k)$*, in the sense of Definition 4.2 coincides with the topological index* $\mathrm{ind_t} D$ *of* D *in the sense of Definition 4.3.* (*This is valid in both the real and complex cases.*)

Proof. First consider the complex case. In the only interesting case, M is even-dimension and D anti-commutes with a $\mathbb{Z}/2$-grading, so as to locally define a class in K^0. Let $N \cong \operatorname{int} M$ be as in Definition 2.2. By the Kasparov-theoretic proof of the usual index theorem [4, Chapter IX, §24.5], the class of D in $K_0(N) \cong K_0(\operatorname{int} M)$ is the Kasparov product $[\sigma(D)]\widehat{\otimes}_{C_0(T^*M)}\alpha$, where

$$\alpha \in KK\big(C_0(T^*M) \otimes C_0(\operatorname{int} M),\, \mathbb{C}\big)$$

is the canonical class coming from the Dolbeault complex $\overline{\partial}$ of the canonical almost complex structure on T^*M, the Thom isomorphism, and the projection map $T^*M \to M$. (In Blackadar's book the proof is given in the case where $\partial M = \emptyset$, but the case where M has a boundary works the same way, once one notices that D and $\overline{\partial}$ define Kasparov bimodules for $C_0(\operatorname{int} M)$, though not for $C(M)$, since we have not imposed any boundary conditions. See also [16, §8, Theorem 1] for an explanation of how to deal with manifolds with boundary.) Now one can observe that everything in sight is compatible with the $\mathbb{Z}/k$-structure, and so descends to the groupoid algebra. In other words, with notation as in Definition 4.3, we have

$$[D] = [\sigma(D)]\widehat{\otimes}_{C_0(T^*M_\Sigma)}\widehat{\alpha} \in K^0(C^*(M; \mathbb{Z}/k)),$$

where now

$$\widehat{\alpha} \in KK\big(C_0(T^*M_\Sigma) \otimes C^*(M; \mathbb{Z}/k),\, \mathbb{C}\big)$$

is the groupoid-equivariant version of α, and we now view $[\sigma(D)]$ as living in $K^0(T^*M_\Sigma)$.

But now $c_*\colon K^0(C^*(M;\mathbb{Z}/k)) \to K^0(C^*(\mathrm{pt};\mathbb{Z}/k)) \xrightarrow{\cong} K_0(\mathrm{pt};\mathbb{Z}/k)$ can be viewed as Kasparov product with the homomorphism c^* of equation (4.1). So by associativity of the Kasparov product, we compute that

$$\mathrm{ind}_{\mathrm{a}}(D) = [c^*]\widehat{\otimes}_{C^*(M;\mathbb{Z}/k)}[D] = [\sigma(D)]\widehat{\otimes}_{C_0(T^*M_\Sigma)}\Big([c^*]\widehat{\otimes}_{C^*(M;\mathbb{Z}/k)}\widehat{\alpha}\Big).$$

So we just need to identify the right-hand side of this equation with $\mathrm{ind}_{\mathrm{t}}(D)$. However, by Definition 4.3, $\mathrm{ind}_{\mathrm{t}}(D) = \widehat{\iota}_!([\sigma(D)])$, where

$$\widehat{\iota}_!\colon K^0(T^*M_\Sigma) \to K^0(T^*D^{2r}_\Sigma) \cong \widetilde{K}^0(M^{2r}_k)$$

is the push-forward map on K-theory. And examination of the definition of $\widehat{\iota}_!$ in terms of the Thom isomorphism shows it is precisely the Kasparov product with

$$[c^*]\widehat{\otimes}_{C^*(M;\mathbb{Z}/k)}\widehat{\alpha},$$

followed by Kasparov product with a “Poincaré duality” element

$$\delta \in KK(C(\mathrm{pt}), C^*(\mathrm{pt};\mathbb{Z}/k) \otimes C_0(M^{2r}_k))$$

implementing the isomorphism $K^0(C^*(\mathrm{pt};\mathbb{Z}/k)) \xrightarrow{\cong} K_0(\mathrm{pt};\mathbb{Z}/k)$. The proof in the real case follows exactly the same outline, except that one has to use the Real structure of the cotangent bundle, i.e., replace $C_0(T^*M)$ by $\{f \in C_0(T^*M) \mid f(\tau(x)) = \overline{f(x)}\}$, where τ is the involution on T^*M that is multiplication by -1 on each fiber. □

DEFINITION 4.5. **(the analytic η-index)** Let $(M, \phi\colon \partial M \xrightarrow{\cong} \beta M \times \eta)$ be a closed η-manifold. We let i be the dimension of M and assume M is equipped with a spin structure inducing a product spin structure on $\partial M^{i-1} \cong \beta M^{i-2} \times S^1$, coming from some spin structure on βM and the *non-bounding* spin structure on S^1. We give M a Riemannian structure compatible with its Σ-structure. For simplicity, we'll first suppose $i = \dim M$ is even and consider D_E, the complex Dirac operator on M with coefficients in some auxiliary bundle E (whose restriction to ∂M is pulled back from βM). The operator D_E is defined using a Hermitian connection on E compatible with the η-manifold structure near ∂E. Since D_E is translation-invariant in a neighborhood of ∂M, it extends to an operator on the open manifold $N = M \cup_{\partial M} \partial M \times [0,\infty) \cong \mathrm{int}\, M$. Thus we have a Kasparov class $[D_E] \in K_0(N)$. (Since N is noncompact, this is to be interpreted as $KK(C_0(N), \mathbb{C})$.) Let $c\colon N \to \mathbb{C}$ be the "collapse" map collapsing M to $0 \in \mathbb{C}$ and sending $\beta M \times S^1 \times [0,\infty)$ first to $S^1 \times [0,\infty)$ (by collapsing the βM factor) and then to $\mathbb{C}$ by means of "polar coordinates" ($(e^{i\theta}, t) \mapsto te^{i\theta}$). We define the *analytic η-index* $\mathrm{ind}_{\mathrm{a}}(D_E)$ of D_E to be $c_*([D_E])$, or in other words the Kasparov product of $[D_E] \in KK(C_0(N), \mathbb{C})$ with the class of the homomorphism

$$c^*\colon\ C_0(\mathbb{C}) \hookrightarrow C_0(N).$$

It takes its values in $K_0(\mathbb{C}) \cong \mathbb{Z}$.

Next, we consider the case of the $\mathcal{C}\ell_i$-linear real Dirac operator with coefficients in a real vector bundle E, whose restriction to ∂M again comes from a bundle E_1 over βM. If we use real instead of complex K-theory, the same procedure as in the complex case gives a KO-valued analytic index $\mathrm{ind}_{\mathrm{a}}(D_E) \in KO_i(\mathbb{C}) \cong KO_{i-2}(\mathrm{pt})$.

THEOREM 4.6 (η-manifold index theorem). *Let $(M, \phi\colon\ \partial M \xrightarrow{\cong} \beta M \times \eta)$ be a closed spin (or spin^c) η-manifold of even dimension, and let E be a vector bundle on M whose restriction to ∂M is pulled back from a bundle E_1 on βM. Fix a Riemannian structure on M compatible with its Σ-structure, and let D_E be the Dirac operator on M with coefficients in E, computed with respect to a Hermitian connection on E whose restriction to a neighborhood of the boundary is pulled back from βM. Then $\mathrm{ind}_{\mathrm{a}}(D_E) = \mathrm{ind}\Big((D_{\beta M})_{E_1}\Big)$, the index (in complex K-theory) of the Dirac operator on βM with coefficients in E_1, which in turn is computed by applying the usual Atiyah-Singer Theorem on βM.*

Proof. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & C(\beta M) \otimes C(S^1) \otimes C_0(\mathbb{R}) \arrow[r] & C_0(N) \arrow[r] & C(M) \arrow[r] & 0 \arrow[d, equal] \\
0 \arrow[r] & C_0(\mathbb{C} \smallsetminus \{0\}) \arrow[r] \arrow[u, hook] & C_0(\mathbb{C}) \arrow[r] \arrow[u, hook] & C(\{0\}) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

This induces a diagram

$$\begin{tikzcd}
K_0(M) \arrow[r] \arrow[d] & K_0(N) \arrow[r] \arrow[d] & K_0(\beta M \times S^1 \times \mathbb{R}) \arrow[d] \\
K_0(\mathrm{pt}) \arrow[r] & K_0(\mathbb{C}) \arrow[r] & K_0(\mathbb{C} \smallsetminus \{0\})
\end{tikzcd}$$

with $[D_E] \in K_0(N)$ mapping by the upper right horizontal arrow to the class of its restriction to the open subset $\beta M \times S^1 \times \mathbb{R}$. Now the bottom row of this diagram is part of the exact sequence

$$\begin{aligned}
&K_1(\mathbb{C}) = 0 \to K_1(\mathbb{C} \smallsetminus \{0\}) \cong \mathbb{Z} \\
&\to K_0(\mathrm{pt}) = \mathbb{Z} \to K_0(\mathbb{C}) \cong \mathbb{Z} \to K_0(\mathbb{C} \smallsetminus \{0\}) \cong \mathbb{Z} \to K_{-1}(\mathrm{pt}) = 0.
\end{aligned}$$

From this we see that the map $K_0(\mathbb{C}) \to K_0(\mathbb{C}\smallsetminus\{0\})$ is an isomorphism, and $K_0(\mathrm{pt}) \to K_0(\mathbb{C})$ is the 0-map. It follows that $\mathrm{ind}_{\mathrm{a}}([D_E])$ can be identified with the image of the restriction of D_E to $\beta M \times S^1 \times \mathbb{R}$ in the group $K_0(\mathbb{C} \smallsetminus \{0\}) \cong \mathbb{Z}$. But by the assumptions on D, this restricted operator splits as the external product

$$\left[(D_{\beta M})_{E_1}\right] \boxtimes [D_{S^1}] \boxtimes [D_{\mathbb{R}}],$$

and maps simply to $k \boxtimes [D_{S^1}] \boxtimes [D_{\mathbb{R}}]$ in $K_0(\mathbb{C}\smallsetminus\{0\}) = K_0(\mathrm{pt} \times S^1 \times \mathbb{R})$, where k is the image of $\left[(D_{\beta M})_{E_1}\right]$ in $K_0(\mathrm{pt})$. But this in turn is just $\mathrm{ind}\,(D_{\beta M})_{E_1}$, while $[D_{S^1}] \boxtimes [D_{\mathbb{R}}]$ is the canonical generator of $K_0(S^1 \times \mathbb{R})$. □

REMARK 4.7. This theorem is in many respects unsatisfactory, since it ignores what happens on $\mathrm{int}\, M$, and also since it fails to take advantage of the free S^1-action on ∂M. One seemingly obvious alternative would be to work with the $\mathbb{R}$-action on N which is trivial on M and which is defined on $\beta M \times S^1 \times [0, \infty)$ by the formula

$$t \cdot (x, e^{i\theta}, s) = (x, e^{i(\theta+ts)}, s).$$

This captures all the η-structure on M, but unfortunately it leads to exactly the same index invariants as the ones we've already defined, because of the fact that the forgetful map $KK^{\mathbb{R}} \to KK$ is an isomorphism ([16], §5, Theorem 2). Another possibility would be to use the S^1-action on N'. (There is no continuous S^1-action on N itself.) Let $G = S^1$ and $R = R(G) = \mathbb{Z}[t,\, t^{-1}]$. We still hope to define out of D_E a class in $KK^G(C_0(N'), \mathbb{C})$. However, existence of such a class seems to be a delicate matter since there is no obvious reason why the restriction of D_N (initially defined on C^∞ spinors on all of N) to those spinors whose restriction to $\partial M = \beta M \times S^1$ is constant in the S^1-factor should be essentially self-adjoint. But if this were the case, or if one could substitute some suitably modified operator, it should define a G-equivariant Kasparov class on N'. Then the *analytic η-index* $\mathrm{ind}_{\mathrm{a}}(D_E)$ of D_E would again be defined to be $c_*([D_E])$, or in other words the Kasparov product of $[D_E] \in KK^G(C_0(N'), \mathbb{C})$ with the class of the G-equivariant homomorphism

$$c^*\colon\ C_0(\mathbb{C}) \hookrightarrow C_0(N').$$

It would take its values in $K_0^G(\mathbb{C}) \cong R = \mathbb{Z}[t,\, t^{-1}]$. (See Proposition 3.3.)

Note that if $\mathfrak{p}$ denotes the augmentation ideal $(t-1)$ of $R = \mathbb{Z}[t,\, t^{-1}]$, then by the Localization Theorem ([24]; see also [22], §3 for slight generalizations) $K_0^G(N')_{\mathfrak{p}} \cong K_0^G(N'^G)_{\mathfrak{p}} \cong K_0(M_\Sigma) \otimes R_{\mathfrak{p}}$ and $K_0^G(\mathbb{C})_{\mathfrak{p}} \cong K_0(\mathbb{C}^G)_{\mathfrak{p}} \cong K_0(\mathrm{pt}) \otimes R_{\mathfrak{p}}$, and c_* localized at $\mathfrak{p}$ really can be identified with the result of the collapse map $M_\Sigma \to \mathrm{pt}$. Thus, after localizing, we would recover the original "naive" notion of index theory on a Σ-manifold, where an elliptic operator gives a class in $K_0(M_\Sigma)$ and the index is obtained via the collapse map to a point.

THEOREM 4.8 (real η-manifold index theorem). *Let (M, ϕ) be a closed spin η-manifold of dimension i, and let E be a real vector bundle on M whose restriction to ∂M is pulled back from a bundle E_1 on βM. Fix a Riemannian structure on M compatible with its Σ-structure, and let D_E be the Dirac operator on M with coefficients in E, computed with respect to a connection on E whose restriction to a neighborhood of the boundary is pulled back from βM. Then the real analytic index of D_E is given by*

$$\mathrm{ind}_{\mathrm{a}}(D_E) = \mathrm{ind}\left((D_{\beta M})_{E_1}\right) \in KO_{i-2}(\mathrm{pt})$$

the index of the real Dirac operator on βM with coefficients in E_1, which in turn is computed by applying the usual (*real*) *Atiyah-Singer Theorem* (*see* [17], Chapter III, §16) *on βM.*

Proof. The proof is almost identical to that of Theorem 4.6. We only indicate the differences:

1. The only case where the map $KO_i(\mathrm{pt}) \to KO_i(\mathbb{C}) \cong KO_{i-2}(\mathrm{pt})$ could possibly be non-zero is when $i \equiv 4 \mod 8$. In this case we have the exact sequence

$$\begin{aligned} KO_4(\mathrm{pt}) \cong \mathbb{Z} \to KO_4(\mathbb{C}) \cong KO_2(\mathrm{pt}) \cong \mathbb{Z}/2 \\ \to KO_4(\mathbb{C} \smallsetminus \{0\}) \cong KO_3(S^1) \xrightarrow{\partial} KO_3(\mathrm{pt}) = 0, \end{aligned}$$

which again shows that the map $KO_4(\mathrm{pt}) \to KO_4(\mathbb{C})$ is zero. So in all cases, $KO_i(\mathbb{C})$ injects into $KO_i(\mathbb{C} \smallsetminus \{0\}) \cong KO_{i-i}(S^1) \cong KO_{i-i}(\mathrm{pt}) \oplus KO_{i-2}(\mathrm{pt})$, in fact as a direct summand.

2. In the real case, the distinction between the two spin structures on S^1 becomes relevant. Since we are using the non-bounding spin structure, $[D_{S^1}] \in K_1(S^1) \cong \mathbb{Z}$ from the proof of Theorem 4.6 has to be replaced by $[D_\eta] \in KO_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Note that $[D_\eta]$ projects to the generator of $\mathbb{Z}/2$ in $KO_1(\mathrm{pt})$, while $[D_{S^1}]$, the Dirac operator for the bounding spin structure, projects to 0 in this factor. But the two Dirac classes have the same projection in $\mathbb{Z} = \widetilde{KO}_1(S^1) \cong KO_0(\mathrm{pt})$, since they differ by the action of $H^1(S^1; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Thus the distinction between $[D_{S^1}]$ and $[D_\eta]$ turns out not to matter after all, as both $[D_{S^1}] \boxtimes [D_{\mathbb{R}}]$ and $[D_\eta] \boxtimes [D_{\mathbb{R}}]$ project to the same thing in the image of $KO_i(\mathbb{C})$ in $KO_i(\mathbb{C} \smallsetminus \{0\}) \cong KO_i(S^1 \times \mathbb{R})$. □

5. Applications to positive scalar curvature

In this section we illustrate the use of the index theorems of Section 4 by reproving some of the results of [6] on positive scalar curvature. First, a simple definition:

DEFINITION 5.1. Let $(M, \phi\colon \partial M \to \beta M \times P)$ be a manifold with singularities $\Sigma = (P)$, as in Definition 2.1. We assume P is equipped with a standard scalar-flat metric. (In our cases, P will be S^1 or $\mathbb{Z}/k$, so this will simply be the usual metric on P.) A *metric of positive scalar curvature* on M means a Riemannian metric on M which in a collar neighborhood of the boundary diffeomorphic to $\beta M \times P \times [0, \varepsilon)$ is a product metric of the form

$$\begin{gathered}(\text{metric on } \beta M) \times (\text{standard metric on } P)\\ \times (\text{standard metric on } [0, \varepsilon))\end{gathered}$$

and which has positive scalar curvature everywhere. Note that since $P \times [0, \varepsilon)$ is scalar-flat, existence of such a metric implies that βM admits a metric of positive scalar curvature.

The following problem, first treated in [6], now arises:

QUESTION 5.2. Suppose βM admits a metric of positive scalar curvature. Then does M admit a metric of positive scalar curvature in the sense of Definition 5.1?

In this regard we have the following result:

THEOREM 5.3. *Let M^n be a closed spin $\mathbb{Z}/k$-manifold. Then if M admits a metric of positive scalar curvature in the sense of Definition 5.1, the analytic index of the Dirac operator of M* (*in the sense of Definition 4.2*) *must vanish in $K_n(\mathrm{pt}; \mathbb{Z}/k)$ or $KO_n(\mathrm{pt}; \mathbb{Z}/k)$.*

Proof. Assume M is a $\mathbb{Z}/k$-manifold with a metric of positive scalar curvature, and form the (complex or $\mathcal{C}\ell_n$-linear real) Dirac operator D with respect to this particular choice of metric. Let N be M with a half-infinite cylinder attached, as in Definition 2.2. Since N is complete, the Lichnerowicz identity $D^2 = \nabla^*\nabla + \frac{s}{4}$ (see [17], Chapter II, Theorem 8.8) is an equality of self-adjoint operators. Here s is the scalar curvature function of N, and since M is compact and s is positive on M and translation-invariant on $\partial M \times [0, \infty)$, s is uniformly bounded below on N by a positive constant. This implies the partial isometry part U of the polar decomposition of D is unitary. Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ be the Hilbert space of L^2-spinors, on which D acts. Note that U is of the form $\begin{pmatrix} 0 & U_0^* \\ U_0 & 0 \end{pmatrix}$, where U_0 is a unitary operator from $\mathcal{H}_0$ onto $\mathcal{H}_1$. The

Kasparov class $[D]$ is defined by $\mathcal{H}$, U, the action of $C^*(M;\mathbb{Z}/k)$, and perhaps an additional action of a Clifford algebra.

Now we claim that $\mathrm{ind}_{\mathrm{a}} D = 0$. Let $A = C^*(\mathrm{pt};\mathbb{Z}/k) \cong \{f \in C_0([0,\infty), M_k) \mid f(0)$ a multiple of $I_k\}$, and let B be the larger unital algebra $\{f \in C_0([0,\infty], M_k) \mid f(0)$ a multiple of $I_k\}$. (Scalars here may be either real or complex, depending on the context.) Then B also acts on $\mathcal{H}$ and $(B, \mathcal{H}, U)$ defines a class in $K^{-n}(B)$, i.e., our class in $K^{-n}(A)$ lies in the image of a class in $K^{-n}(B)$ defined by the same operator U. The reason is that C^∞ functions φ on N that are eventually constant on each cylinder $\beta M \times [0,\infty)$ have vanishing gradient in a neighborhood of infinity, and thus their commutator with D has compact support. On the other hand, $U = D|D|^{-1}$ and $|D|^{-1}$ is pseudodifferential of order -1. From this one can deduce $[U,\varphi] = [D,\varphi]|D|^{-1} + D[|D|^{-1},\varphi]$ is compact; the proof of this is quite similar to Proposition 3.3 in [11]. Here are the details. The first term, $[D,\varphi]|D|^{-1}$, is the product of a negative-order pseudodifferential operator with multiplication by a function of compact support, so this is compact. The second term is pseudodifferential of negative order and hence bounded; we want to show it is compact. We have (using an identity from the proof of [4], Proposition 17.11.3):

$$D[|D|^{-1},\varphi] = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2} D\left[(D^2+\lambda)^{-1},\varphi\right] d\lambda, \tag{5.1}$$

but

$$[(D^2+\lambda)^{-1},\varphi] = -(D^2+\lambda)^{-1}[(D^2+\lambda),\varphi](D^2+\lambda)^{-1}$$

and $[(D^2+\lambda),\varphi] = [D,\varphi]D + D[D,\varphi]$ has compact support, hence $[(D^2+\lambda),\varphi](D^2+\lambda)^{-1}$ is compact, and then

$$D\left[(D^2+\lambda)^{-1},\varphi\right] = -D(D^2+\lambda)^{-1}\left[(D^2+\lambda),\varphi\right](D^2+\lambda)^{-1}$$

is compact. Then by equation (5.1), the remaining term in $[U,\varphi]$ is also compact. Since $U^2 = 1$ and $U = U^*$, there is nothing to check as far as the other axioms for a Kasparov bimodule are concerned, so $(B,\mathcal{H},U)$ defines a class in $K^{-n}(B)$. Furthermore, we have a short exact sequence

$$0 \to C_0((0,\infty], M_k) \to B \to C(\mathrm{pt}) \to 0,$$

with the ideal $C_0((0,\infty], M_k)$ contractible, and thus $K^{-n}(B) \cong K_n(\mathrm{pt})$. The map $B \to C(\mathrm{pt})$ is split by the inclusion of scalar multiples of the identity, under which the class of $(B,\mathcal{H},U)$ pulls back to the class of $(\mathcal{H},U)$. So via the isomorphism $K^{-n}(B) \cong K_n(\mathrm{pt})$, we see that the class of $(B,\mathcal{H},U)$ can be identified with class of U (perhaps with some auxiliary Clifford algebra action, if we're in the case of the $\mathcal{C}\ell_n$-linear real Dirac operator), which vanishes, since U is unitary. Hence $c_*([D]) = 0$. □

This gives another proof of the "only if" direction of the following theorem from [6]:

THEOREM 5.4 (Botvinnik [6]). *Let M^n be a closed spin $\mathbb{Z}/2$-manifold, with $n \geq 6$, and assume M and βM are connected and simply connected. Then M admits a metric of positive scalar curvature in the sense of Definition 5.1 if and only if the image $\alpha_\Sigma(M)$ of the canonical class defined by the spin structure in $KO_n(M;\mathbb{Z}/2)$ vanishes in $KO_n(\mathrm{pt};\mathbb{Z}/2)$.*

REMARK 5.5. Note that the obstruction $\alpha_\Sigma(M) \in KO_n(M;\mathbb{Z}/2)$ of Theorem 5.4 includes within it the obstruction to existence of a positive scalar curvature metric on βM. Indeed, since $n \geq 6$, $\dim \beta M \geq 5$, so we know from [25] that βM admits a metric of positive scalar curvature if and only if the usual index invariant $\alpha(\beta M) \in KO_{n-1}(\mathrm{pt})$ vanishes. Now the existence of spin structure on M with $\partial M \equiv \beta M \amalg \beta M$ implies that the class of βM must be a 2-torsion class in $\Omega^{\mathrm{Spin}}_{n-1}$, which forces the $\widehat{A}$-genus of βM to vanish if $n \equiv 1 \pmod 4$. On the other hand, we claim that under the exact sequence

$$\cdots \longrightarrow KO_n(\mathrm{pt}) \xrightarrow{2} KO_n(\mathrm{pt}) \longrightarrow KO_n(\mathrm{pt};\mathbb{Z}/2) \xrightarrow{\partial} KO_{n-1}(\mathrm{pt}) \xrightarrow{2} \cdots,$$

$\partial(\alpha_\Sigma(M)) = \alpha(\beta M)$, so $\alpha_\Sigma(M) \neq 0$ if $\alpha(\beta M)$ is a non-zero 2-torsion class. For example, if βM is an exotic 9-sphere with $\alpha(\beta M)$ a non-zero 2-torsion class in Ω^{Spin}_9, then $\beta M \amalg \beta M$ bounds a spin 10-manifold M which can be given a $\mathbb{Z}/2$-manifold structure, and $\alpha_\Sigma(M) \neq 0$.

To check this, observe that by Theorem 4.4, the topological invariant $\alpha_\Sigma(M)$ coincides with $\mathrm{ind}_{\mathrm{a}}(D)$, D the Dirac operator on M (or more exactly on N, the manifold with cylinders attached). Let $[D] \in KO^{-i}(C^*(M;\mathbb{Z}/2))$ be the associated class. Via the exact sequence in Proposition 2.3, this restricts to $[D_{\beta M}]$, the class of the Dirac operator on βM, in

$$KO^{-i}(C_0(\mathbb{R}) \otimes C(\beta M) \otimes M_2) \cong KO_{i-1}(\beta M).$$

Similarly, Theorem 5.7 below is related to the "only if" direction of the following theorem from [6]:

THEOREM 5.6 (Botvinnik [6]). *Let M^n be a closed spin η-manifold, with $n \geq 7$, and assume M and βM are connected and simply connected. Then M admits a metric of positive scalar curvature in the sense of Definition 5.1 if and only if the image $\alpha_\Sigma(M)$ of the canonical class defined by the spin structure in $KU_n(M)$ vanishes in $KU_n(\mathrm{pt})$.*

THEOREM 5.7. *Let (M^n, ϕ) be a closed spin η-manifold, and suppose M admits a metric of positive scalar curvature in the sense of Definition 5.1. Then the analytic η-index $\mathrm{ind}_{\mathrm{a}}(D) \in KO_n(\mathbb{C}) \cong KO_{n-2}(\mathrm{pt})$ of Dirac operator on M (as defined in Definition 4.5) must vanish.*

Proof. By the index theorem 4.8, $\mathrm{ind}_{\mathrm{a}}(D)$ may be identified with the index of the real Dirac operator on βM, which is an obstruction to positive scalar curvature on βM [12]. However, a metric of positive scalar curvature on M must by definition restrict to a metric of positive scalar curvature on $\partial M \cong \beta M \times S^1$ which is a product of a metric on βM with the standard flat metric on S^1. So such a metric can only exist when βM admits a metric of positive scalar curvature. □

REMARK 5.8. One can also give another proof of Theorem 5.7 along the lines of the proof of Theorem 5.3. In addition, the argument used to prove Theorem 5.3 can be extended to give an interpretation of the $\mathbb{Z}/k$-index of the Dirac operator in a more general context.

THEOREM 5.9. *Let M^n be a closed spin $\mathbb{Z}/k$-manifold, equipped with a $\mathbb{Z}/k$-metric restricting on βM to a metric of positive scalar curvature. Let $N = M \cup_{\partial M} \partial M \times [0, \infty)$ and let D be the $\mathcal{C}\ell_n$-linear real Dirac operator on N. Then D (acting on the $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H}$ of L^2 sections of the appropriate bundle of free right $\mathcal{C}\ell_n$-modules over N) has finite-dimensional kernel, and the $\mathbb{Z}/k$-index of D is the mod-k reduction of the $KO(\mathrm{pt})_n$-valued index of D (computed from the kernel of D over N, viewed as a graded $\mathcal{C}\ell_n$-module).*

Proof. Because N has *uniformly* positive scalar curvature on the ends $\beta M \times [0, \infty)$, the spectrum of D is bounded away from 0 on the complement of a finite-dimensional subspace of $\mathcal{H}$ ([10], Theorem 3.2). Then if U is the partial isometry part of the polar decomposition of D, U is Fredholm and the $\mathbb{Z}/k$-index of D is defined by the triple $(C^*_{\mathbb{R}}(\mathrm{pt}; \mathbb{Z}/k), \mathcal{H}, U)$. As in the proof of 5.3, this Kasparov class is in the image of $KO^{-n}(B)$, where B is a suitable unital extension of $C^*_{\mathbb{R}}(\mathrm{pt}; \mathbb{Z}/k)$ with the same KO-theory as the scalars. The class in $KO^{-n}(B)$ that maps to the $\mathbb{Z}/k$-index of D is defined by $(U, \mathcal{H})$ (together with the $\mathcal{C}\ell_n$-action), and the map

$$KO^{-n}(B) \to KO^{-n}(C^*_{\mathbb{R}}(\mathrm{pt}; \mathbb{Z}/k))$$

can be identified with the map $KO_n(\mathrm{pt}) \to KO_n(\mathrm{pt}; \mathbb{Z}/k)$, which is simply reduction mod k. So $\mathrm{ind}_{\mathrm{a}} D$ is the mod k reduction of the $KO(\mathrm{pt})_n$-valued index of D. □

References

1. M. F. Atiyah, *K-theory and reality*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 367–386. MR 34 #6756
2. M. F. Atiyah and G. B. Segal, *Equivariant K-theory and completion*, J. Differential Geometry **3** (1969), 1–18. MR 41 #4575
3. Nils Andreas Baas, *On bordism theory of manifolds with singularities*, Math. Scand. **33** (1973), 279–302 (1974). MR 49 #11547b
4. Bruce Blackadar, *K-theory for operator algebras*, second ed., Cambridge University Press, Cambridge, 1998. MR 99g:46104
5. Boris Botvinnik, *Manifolds with singularities and the Adams-Novikov spectral sequence*, Cambridge University Press, Cambridge, 1992. MR 93h:55002
6. ———, *Manifolds with singularities accepting a metric of positive scalar curvature*, Geom. Topol. **5** (2001), 683–718 (electronic). MR 1 857 524
7. Alain Connes, *Noncommutative geometry*, Academic Press Inc., San Diego, CA, 1994. MR 95j:46063
8. Daniel S. Freed, *$\mathbb{Z}/k$-manifolds and families of Dirac operators*, Invent. Math. **92** (1988), no. 2, 243–254. MR 89f:58127
9. Daniel S. Freed and Richard B. Melrose, *A mod k index theorem*, Invent. Math. **107** (1992), no. 2, 283–299. MR 93c:58212
10. Mikhael Gromov and H. Blaine Lawson, Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. (1983), no. 58, 83–196 (1984). MR 85g:58082
11. Nigel Higson, *An approach to $\mathbb{Z}/k$-index theory*, Internat. J. Math. **1** (1990), no. 2, 189–210. MR 91h:58109
12. Nigel Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55. MR 50 #11332
13. Pierre Julg, *K-théorie équivariante et produits croisés*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 13, 629–632. MR 83b:46090
14. J. Kaminker and K. P. Wojciechowski, *Index theory of $\mathbb{Z}/k$ manifolds and the Grassmannian*, Operator algebras and topology (Craiova, 1989), Longman Sci. Tech., Harlow, 1992, pp. 82–92. MR 93j:58127. Note: The published version of the paper is missing two pages. They are available from the authors at `www.math.iupui.edu`.
15. G. G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 3, 571–636, 719 (Russian), transl. in Math. USSR Izv. **16** (1981), 513–572. MR 81m:58075
16. ———, *K-theory, group C^*-algebras, and higher signatures (conspectus)*, Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), Cambridge Univ. Press, Cambridge, 1995, pp. 101–146. MR 97j:58153
17. H. Blaine Lawson, Jr. and Marie-Louise Michelsohn, *Spin geometry*, Princeton University Press, Princeton, NJ, 1989. MR 91g:53001

18. John W. Morgan and Dennis P. Sullivan, *The transversality characteristic class and linking cycles in surgery theory*, Ann. of Math. (2) **99** (1974), 463–544. MR 50 #3240
19. Alan L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Birkhäuser Boston Inc., Boston, MA, 1999. MR 2001a:22003
20. Jean Renault, *A groupoid approach to C^*-algebras*, Springer, Berlin, 1980. MR 82h:46075
21. Jonathan Rosenberg, *Review of "Elements of KK-theory" by K. K. Jensen and K. Thomsen*, Bull. Amer. Math. Soc. (N. S.) **28** (1993), no. 2, 342–347.
22. ———, *The G-signature theorem revisited*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), Amer. Math. Soc., Providence, RI, 1999, pp. 251–264. MR 2000h:58047
23. Claude Schochet, *The Kasparov groups for commutative C^*-algebras and Spanier-Whitehead duality*, Operator theory: operator algebras and applications, Part 2 (Durham, NH, 1988), Amer. Math. Soc., Providence, RI, 1990, pp. 307–321. MR 92e:46138
24. Graeme Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 129–151. MR 38 #2769
25. Stephan Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **136** (1992), no. 3, 511–540. MR 93i:57033
26. D. P. Sullivan, *Triangulating and smoothing homotopy equivalences and homeomorphisms: geometric topology seminar notes*, The Hauptvermutung book, Kluwer Acad. Publ., Dordrecht, 1996, pp. 69–103. MR 98c:57027
27. Weiping Zhang, *On the mod k index theorem of Freed and Melrose*, J. Differential Geom. **43** (1996), no. 1, 198–206. MR 97k:58157