RANK RIGIDITY OF EUCLIDEAN POLYHEDRA

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Abstract. A Euclidean polyhedron (a simplicial complex whose simplices are Euclidean) of nonpositive curvature (in the sense of Alexandrov) has rank ≥ 2 if every finite geodesic segment is a side of a flat rectangle. We prove that if a three-dimensional, geodesically complete, simply connected Euclidean polyhedron X of rank ≥ 2 and of nonpositive curvature admits a cocompact and properly discontinuous group of isometries, then X is either a Riemannian product or a thick Euclidean building of type $\widetilde{A}_3$ or $\widetilde{B}_3$.

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1. Introduction

Let X be a simply connected, complete, geodesic space of nonpositive curvature in the sense of Alexandrov [Bal95]; in other words, X is a *Hadamard space.* Assume that X admits a properly discontinuous and cocompact group Γ of isometries. If X is a Riemannian manifold and every geodesic in X bounds a flat strip, then X is either a Riemannian product or a symmetric space of higher rank, see [Bal95]. In the general case we also expect that X belongs to a relatively short list of model spaces if there is "enough" zero curvature in X.

Similarly to Riemannian manifolds it is natural to introduce a notion of rank for X and establish rank rigidity by giving a complete classification of spaces of rank ≥ 2. There are several ways of defining that X has rank ≥ 2:

- every geodesic belongs to a flat plane;
- every geodesic bounds a flat half plane;
- every geodesic is a side of a flat strip;
- every finite geodesic segment is a side of a flat rectangle.

Obviously the further down the list the weaker the notion. In the Riemannian case all four notions are equivalent. In this paper we use the last property and say that X has *rank* ≥ 2 if every finite geodesic segment is a side of a flat rectangle, otherwise we say that X has *rank 1.* In [BB95] we proved that if X is a two-dimensional, geodesically complete polyhedron with a piecewise smooth metric of nonpositive curvature, then either X has rank 1 and contains a Γ-closed hyperbolic geodesic or X has rank 2 and is either a product of two trees or a thick Euclidean building of type $\widetilde{A}_2$, $\widetilde{B}_2$ or $\widetilde{G}_2$ (the notion of rank 1 that we use here differs slightly from the notion of rank 1 in [BB95]). The following theorem is the main result of this paper.

1.1. THEOREM. *Let X be a three-dimensional, geodesically complete, simply connected Euclidean polyhedron of nonpositive curvature admitting a properly discontinuous and cocompact group of isometries. If X has rank ≥ 2, then X is either a Riemannian product or a thick Euclidean building of type $\widetilde{A}_3$ or $\widetilde{B}_3$.*

The regularity assumption of geodesic completeness implies that X has no boundary. The following result shows that the cocompactness assumption in Theorem 1.1 is necessary.

1.2. THEOREM. *There exists a three-dimensional, geodesically complete, simply connected Euclidean polyhedron of nonpositive curvature in which every geodesic lies in a flat plane but which is not a product or a thick Euclidean building.*

If X has rank ≥ 2, the links in X are two-dimensional spherical polyhedra of diameter and injectivity radius π. By [BB98], there are three types of such polyhedra: (possibly reducible) spherical buildings, spherical joins with equator of diameter $> \pi$ and hemispherexes (the precise definitions are given in Section 2). In Section 3 we show that if one link in X is a hemispherex, then X has rank 1. In Section 4 we show that if one link in X is a spherical join with equator of diameter $> \pi$, then X is a product of a tree and a two-dimensional Euclidean polyhedron.

B.Kleiner (1993, unpublished) and A.Lytchak [Lyt98] proved that if all links of X are spherical buildings, then X is a Euclidean building. Together with our results from Sections 3 and 4 this implies Theorem 1.1 since three-dimensional, thick, irreducible Euclidean buildings are of type $\widetilde{A}_3$ or $\widetilde{B}_3$.

2. Preliminaries

A topological space is called a *polyhedron* if it admits a triangulation. A polyhedron with a length metric is called Euclidean (respectively, spherical) if it admits a triangulation into Euclidean (respectively, spherical) simplices. Here a Euclidean (respectively, spherical) k-simplex is a k-simplex A such that A with the induced length metric is isometric to the intersection of $k+1$ closed half spaces in $\mathbb{R}^k$ (respectively, closed hemispheres in S^k) in general position.

Let (X,d) be a locally finite Euclidean polyhedron, $x \in X$ and let A be a closed k-simplex containing x. View A as a subset of $\mathbb{R}^k$ and set S_xA to be the set of unit tangent vectors ξ at x such that a nontrivial line segment with initial direction ξ is contained in A. If $B \subset A$ is another closed simplex containing x, then naturally $S_xB \subset S_xA$. We define the *link* S_xX of X at x by

$$S_xX = \bigcup_{A \ni x} S_xA\,,$$

where the union is taken over all closed simplices containing x. If the maximal dimension of a simplex adjacent to x is n then S_xX has dimension $n-1$. Angles in S_xA induce a natural length metric d_x on S_xX which turns it into a spherical polyhedron. For $\xi, \eta \in S_xX$ define

$$\angle(\xi,\eta) = \min(d_x(\xi,\eta),\pi)\,.$$

For every $x \in X$ there is a neighborhood U of x with polar coordinates (ξ, s), $\xi \in S_xX$, $0 \leq s \leq \delta$, centered at x and such that

$$d^2((\xi,s),(\eta,t)) = s^2 + t^2 - 2st\cos\angle(\xi,\eta)\,.$$

In other words, the δ-neighborhood of x in X is isometric to the *Euclidean cone* of radius δ over S_xX, where x corresponds to the apex of the cone. Similar definitions and constructions apply to spherical polyhedra.

A curve $\gamma : I \to X$ is a *geodesic* if it has constant speed and is locally distance minimizing. It it easy to show that a curve $\gamma : [a, b] \to X$ with constant speed is a geodesic if and only if there is a subdivision $a = t_0 < t_1 < ... < t_m = b$ such that

1. $\gamma([t_{j-1}, t_j])$ is contained in a closed simplex $A_j \subset X$ and $\gamma : [t_{j-1}, t_j] \to A_j$ is a standard geodesic segment in A_j, $1 \le j \le m$;
2. $\angle(-\dot\gamma(t_j), \dot\gamma(t_j)) = \pi$, $1 \le j \le m - 1$.

We refer to $-\dot\gamma(t)$ and $\dot\gamma(t)$ as the *incoming* and *outgoing* directions of γ in the link $S_{\gamma(t)}X$. For geodesics γ and σ with $\gamma(s) = \sigma(t) =: x$ we set

$$\angle_x(\gamma, \sigma) = \angle(\dot\gamma(s), \dot\sigma(t)) .$$

We say that X is *geodesically complete* if every geodesic $\gamma : I \to X$ can be extended to a geodesic $\tilde\gamma : \mathbb{R} \to X$. If X is complete as a metric space, then X is geodesically complete if and only if for every $x \in X$ and every $\xi \in S_xX$ there is $\eta \in S_xX$ with $d_x(\xi, \eta) \ge \pi$. If X is geodesically complete then it has no *boundary simplices*, that is, simplices which are adjacent to exactly one simplex of a higher dimension.

The *injectivity radius* $\operatorname{inj} Z$ of a geodesic space Z is the supremum of the set of $r \ge 0$ such that any geodesic segment of length $\le r$ is the unique minimal connection between its ends.

Since locally our Euclidean polyhedron X is a Euclidean cone over the link, it has nonpositive curvature iff the injectivity radius of S_xX is $\ge \pi$ for all $x \in X$. Assume that the rank of X is ≥ 2, let $x \in X$ and let ξ, η be two directions in S_xX which lie at distance $\ge \pi$. There is a geodesic segment passing through x with incoming direction ξ and outgoing direction η. Since this geodesic segment bounds a flat strip, there is a semicircle (of length π) connecting ξ to η. Therefore S_xX has diameter π. Since X is geodesically complete and is locally isometric to a Euclidean cone over S_xX, the space S_xX is also geodesically complete. Hence for $\dim X = 3$ the link $Y = S_xX$ is a geodesically complete compact 2-dimensional spherical polyhedron of diameter and injectivity radius π. Here are examples of geodesically complete compact spherical polyhedra of diameter and injectivity radius π.

Spherical building. If Y is a spherical building then Y carries a natural metric for which the apartments are unit spheres. For this metric the diameter and injectivity radius of Y are π. If X is a Euclidean building of dimension $n \geq 2$ with the natural metric, then every geodesic in X is contained in an isometrically embedded convex Euclidean n-space. The link of a vertex in X is a spherical building of dimension $n-1$ which has injectivity radius and diameter π. An n-dimensional building Z is called *thick* if every $(n-1)$-simplex of Z is adjacent to at least three n-simplices. See [Bro96].

Spherical join. Let X_i be a Euclidean polyhedron, $x_i \in X_i$ and $Y_i = S_{x_i}X_i$, $i = 1, 2$. Then the *spherical join* $Y_1 * Y_2$ is the link of $(x_1, x_2) \in X_1 \times X_2$ with angular distance. Clearly $Y_1 * Y_2$ is a spherical polyhedron of dimension $\dim Y_1 + \dim Y_2 + 1$ and naturally contains Y_1 and Y_2 as subpolyhedra. If X_1 and X_2 are geodesically complete and of nonpositive curvature then $X_1 \times X_2$ has rank ≥ 2 and $Y_1 * Y_2$ is geodesically complete and has injectivity radius and diameter π.

If $\dim X_1 = 1$, then Y_1 is a finite set and $Y_1 * Y_2$ admits the following simple description which is sufficient for this paper. For each $p \in Y_1$ the *spherical cone* C_p over Y_2 with *pole* p is the product $Y_2 \times [0, \pi/2]$ in which $Y_2 \times \{0\}$ is identified with p. The distance d in $Y_1 * Y_2$ is given by

$$\cos d((y, s), (z, t)) = \cos s \cos t + \sin s \sin t \cos \min\{\pi, d_2(y, z)\},$$

where d_2 is the distance in Y_2. The spherical join $Y_1 * Y_2$ is the disjoint union of the spherical cones C_p, $p \in Y_1$, identified along the *equators* $Y_2 \times \{\pi/2\}$.

Hemispherex. A spherical polyhedron is called a *hemispherex* if it is obtained from the unit sphere by attaching unit hemispheres along great hyperspheres so that no pair of antipodal points on the sphere belongs to all hyperspheres. In particular, at least three hemispheres must be attached to the unit sphere to create a hemispherex of dimension greater than 1. A 1-dimensional hemispherex is called a *semicirclex.* It is not difficult to see that the injectivity radius of a hemispherex is π, the diameter of a semicirclex is $> \pi$ and the diameter of a hemispherex of dimension ≥ 2 is π, see Figure 1.

We refer to the initial sphere (circle) of a hemispherex (semicirclex) X as the *central sphere (circle)* of X. The central circle of a semicirclex is distinguished as the set of points x for which the maximum of the distance function from x is π. For all other points it is $> \pi$. In dimension $n > 1$, each essential vertex x of a hemispherex X lies on the central sphere S, the link S_xX is an $(n-1)$-dimensional hemispherex, and S is distinguished recursively by the property that S is tangent to the central sphere of S_xX.

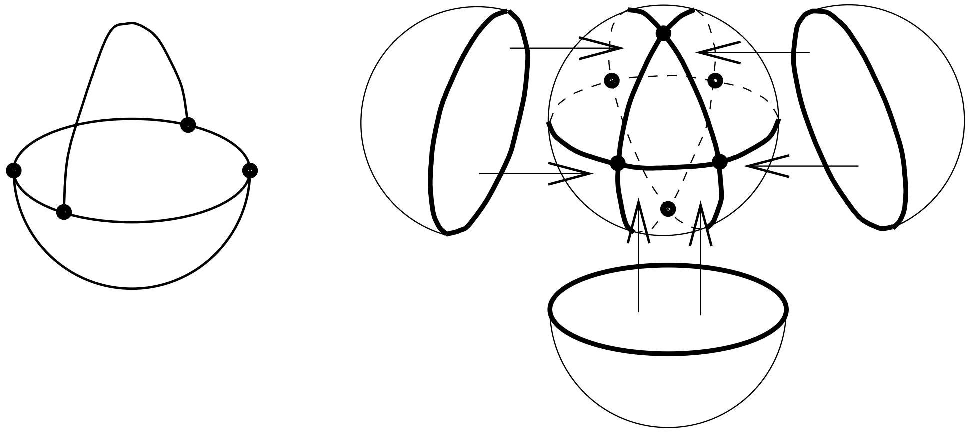

FIGURE 1. The simplest semicirclex and hemispherex.

2.1. THEOREM. [BB98] *Let X be a geodesically complete, compact, 2-dimensional spherical polyhedron of diameter and injectivity radius π. Then X is either*

- a thick spherical building of type A_3 or B_3, or

- a spherical join of a finite set with a metric graph of injectivity radius $\geq \pi$, or

- a hemispherex.

The Euclidean cone $C(H)$ over a hemispherex H of dimension ≥ 2 is a complete simply connected Euclidean polyhedron of nonpositive curvature and rank ≥ 2. This is the only known to us example of a higher rank space of nonpositive curvature which is not a Euclidean building, a symmetric space or a product. Obviously, every isometry of $C(H)$ must fix the cone point and hence is of finite order. In particular, $C(H)$ does not admit compact factors. This explains the necessity of the cocompactness assumption in the main theorem and proves Theorem 1.2.

Let Y be a Hadamard space. Classes of asymptotic geodesic rays in Y form the space $Y(\infty)$ of points at ∞ and the compactification $\overline{Y} = Y \cup Y(\infty)$ has a natural Γ-invariant topology. A closed, convex subset $Z \subset Y$ is *geodesically complete* if every geodesic segment in Z can be extended to a complete geodesic in Z. Closed, convex and geodesically complete subsets Z_1 and Z_2 are *parallel* if $Z_1(\infty) = Z_2(\infty)$.

2.2. PROPOSITION. *Let Y be a Hadamard space and $Z \subset Y$ be a closed, convex and geodesically complete subset. Let Y_Z be the set of points $y \in Y$ for which there is a closed, convex and geodesically complete subset $Z(y)$ parallel to Z and containing y. Then Y_Z is closed* *and convex and is isometric to $C \times Z$, where $C \subset Y_Z$ is closed and convex.*

Proof. Step 1. Let Z_0 and Z_1 be parallel. Denote by P_i be the (nearest point) projection to Z_i, $i = 0, 1$, and let $f(\cdot) = d(\cdot, Z_1)$. Then f is convex and continuous. Let x, x' be distinct points in Z_0 and let $\sigma_0 : \mathbb{R} \to Z_0$ be a complete geodesic through x and x'. Then $\sigma(\pm\infty) \in Z_0(\infty) = Z_1(\infty)$ and hence f is bounded on σ_0. Since f is convex, it is constant on σ_0. Since x and x' were arbitrary, f is constant on Z_0. It also follows that $\sigma_1 = P_1\sigma_0$ is a geodesic in Z_1 which is parallel to σ_0 and that the convex hull of Z_0 and Z_1 is isometric to the product $[0, a] \times Z_0$, where $a = d(Z_0, Z_1)$. In particular P_0P_1 is the identity on Z_0 and P_0 and P_1 preserve distances on Z_1 and Z_0 respectively.

Step 2. Let Z_2 be parallel to Z_1 (and Z_0). Denote by P_2 the (nearest point) projection to Z_2. Let $x \in Z_0$ and $y = P_0P_1P_2x \in Z_0$. Suppose $y \neq x$ and let $\rho : [0, \infty) \to Z_0$ be a geodesic ray through y starting from x. If b is the Busemann function associated to ρ, then $b(x) = 0$ and $b(y) < 0$. By Step 1, $\rho_2 = P_2\rho$ is the geodesic ray in Z_2 starting from P_2x which is asymptotic to ρ, and ρ and ρ_2 are parallel. Hence the Busemann function b_2 associated to ρ_2 coincides with b. Similarly, if b_1 is the Busemann function associated to the geodesic ray $\rho_1 = P_1\rho_2 = P_1P_2\rho$ starting from P_1P_2x and asymptotic to ρ_2 (and ρ), then $b_1 = b_2 = b$. Finally, if b_0 is the Busemann function associated to the geodesic ray $\rho_0 = P_0\rho_1 = P_0P_1P_2\rho$ starting from $y = P_0P_1P_2x$ and asymptotic to ρ_1 (and ρ), then $b_0 = b_1 = b_2 = b$. Since $y = \rho(t)$ for some $t > 0$, we have that $\rho_0(s) = \rho(t + s)$ for all $s > 0$ and hence $b_0 = b + t$ which is a contradiction. Therefore $x = y$ and $P_0P_1P_2$ is the identity on Z_0.

Step 3. Let P_Z be the projection to Z. For $y \in Y_Z$, let P_y denote the (nearest point) projection to $Z(y)$. Fix $z_0 \in Z$ and set $C = \cup_{y \in Y_Z} P_y z_0$. By Step 2, the geodesic segment between any two points $c_1, c_2 \in C$ projects to z_0 under P_Z, and hence C is convex. Clearly C is closed. Denote by P_C the projection to C. Let $y_i \in Y_Z$ and let $a_i = d(Z(y_i), Z)$, $i = 1, 2$. Then, by Step 1, the convex hull of $Z(y_i)$ and Z is isometric to $[0, a_i] \times Z$, and $P_C y_i = P_{y_i} z_0 \in Z(y_i)$, $i = 1, 2$. Similarly, the convex hull of $Z(y_1)$ and $Z(y_2)$ is isometric to $[0, a] \times Z$, where $a = d(Z(y_1), Z(y_2))$. Therefore and by Step 1 and Step 2,

$$\begin{aligned} d^2(y_1, y_2) &= d^2(P_C y_1, P_C y_2) + d^2(y_1, P_{y_1} y_2) \\ &= d^2(P_C y_1, P_C y_2) + d^2(P_Z y_1, P_Z P_{y_1} y_2) \\ &= d^2(P_C y_1, P_C y_2) + d^2(P_Z y_1, P_Z y_2)\,. \end{aligned}$$

□

3. The Case of a Hemispherex

Let X be a three-dimensional, geodesically complete and simply connected Euclidean polyhedron of nonpositive curvature admitting a properly discontinuous and cocompact group Γ of isometries. In this section we show that X has rank one if there is a point $x \in X$ whose link is a hemispherex.

3.1. PROPOSITION. *Let $x \in X$ and suppose that S_xX is a hemispherex. Then there exists an isometry $\varphi \in \Gamma$ such that the unit speed geodesic segment $\sigma : [0, l] \to X$ from x to φx can be extended to a geodesic segment $\widehat{\sigma} : [-\varepsilon, l + \varepsilon] \to X$ that is not a side of a flat rectangle.*

The proof of Proposition 3.1 uses the geodesic flow g^t on X and the g^t-invariant Liouville measure μ constructed in [BB95]. Denote by GX the set of unit speed geodesics $\sigma : \mathbb{R} \to X$ endowed with the compact-open topology. The geodesic flow g^t acts on GX by $g^t(\sigma)(s) = \sigma(s+t)$.

We say that a unit speed geodesic $\sigma : I \to X$ is *regular* if σ does not meet the 1-skeleton $X^{(1)}$ of X. The Liouvile measure μ is concentrated on the set G_RX of complete regular geodesics and is positive on open subsets of G_RX.

A geodesic $\sigma : \mathbb{R} \to X$ is called Γ-*recurrent* if there are sequences $t_n \to \infty$ and $\varphi_n \in \Gamma$ such that $\varphi_n^{-1} g^{t_n}(\sigma) \to \sigma$.

3.2. LEMMA. *(*[BB95]*, Section 3) For any regular geodesic segment $\sigma : [a, b] \to X$, there are complete, Γ-recurrent, regular geodesics σ_n such that $\sigma_n | [a, b] \to \sigma$.* □

3.3. LEMMA. *Let $y \in X$, $\xi, \eta \in S_yX$ and suppose that there is a sequence of complete regular geodesics σ_n such that $\sigma_n | [-\delta, \delta]$ converges to a geodesic segment $\sigma_0 : [-\delta, \delta] \to X$ through y with $\dot{\sigma}_0(0) = \xi$ and $-\dot{\sigma}_0(0) = \eta$. Then for every $\varepsilon > 0$ there is $\varphi \in \Gamma$ such that the geodesic $\sigma : [0, r] \to X$ from y to φy satisfies $d_y(\dot{\sigma}(0), \xi) < \varepsilon$ and $d_{\varphi y}(-\dot{\sigma}(r), \eta) < \varepsilon$.*

Proof. By Lemma 3.2, we can assume that the geodesics σ_n are recurrent. Hence there are $\varphi_n \in \Gamma$ and $t_n \to \infty$ such that $\varphi_n^{-1} g^{t_n} \sigma_n | [-\delta, \delta] \to \sigma_0$ as $n \to \infty$. Set $p_n = \sigma_n(0)$ and $q_n = \sigma_n(t_n)$. Then $d(p_n, y) \to 0$ and $d(q_n, \varphi_n y) = d(\varphi_n^{-1} q_n, y) \to 0$ as $n \to \infty$. Let $\rho_n : [0, r_n] \to X$ be the unit speed geodesic from y to $\varphi_n y$. By convexity, $d(\sigma_n(t), \rho_n(t) \to 0$ uniformly in t. Therefore $d_y(\dot{\rho}_n(0), \xi) \to 0$ and $d_{\varphi_n y}(-\dot{\rho}_n(r_n), \varphi_n \eta) \to 0$ as $n \to \infty$. □

3.4. LEMMA. *Suppose that S_xX is a hemispherex, ξ is a point in the central sphere, η is a point in an open attached hemisphere H and*

$d_x(\xi,\eta)=\pi$. Choose $\delta>0$ such that the δ-neighborhood of x is isometric to the Euclidean cone over S_xX of radius δ. Let $\sigma_0:[-\delta,\delta]\to X$ be the unique geodesic segment with $\dot\sigma_0(0)=\xi$ and $-\dot\sigma_0=\eta$. Then there are complete regular geodesics σ_n such that $\sigma_n|[-\delta,\delta]\to\sigma_0$.

Proof. The equator of H breaks S into two hemispheres, H_1 and H_2. If $\xi\in H_1$, then $S'=H\cup H_1$ is a sphere representing a Euclidean ball $B\subset X$ centered at x of radius δ. Since $X^{(1)}$ intersects B in a finite set of radii, σ_0 can be approximated as claimed. □

3.5. LEMMA. *Suppose that Y is a hemispherex, H is an open attached hemisphere and $\xi\in H$. Let H' be an open attached hemisphere such that $\overline{H}\cap\overline{H}'$ consists of two points. Then there is $\eta\in H'$ lying at distance π to ξ and such that there are exactly two geodesics of length π connecting ξ and η. Their union is a circle of length 2π passing through the two points of the intersection $\overline{H}\cap\overline{H}'$.*

Proof. Let S be the central sphere of Y. The union $Z=S\cup H\cup H'$ is a spherical join whose equator E_Z is the union of a circle (corresponding to S) with two semicircles (corresponding to H and H') and whose poles are the the two points of the intersection $\overline{H}\cap\overline{H}'$. Since every unit speed geodesic in Y entering an open attached hemisphere H'' spends time π in H'', any minimal (of length π) connection from ξ to a point $\eta\in H'$ lies in Z. Now choose $\eta\in H'$ at distance π from ξ and such that the projections of ξ and η to E_Z lie at distance greater than π in E_Z. □

End of proof of Proposition 3.1. Let $S\subset S_xX$ be the central sphere and $E\subset S$ be the union of the equators of the attached hemispheres. Let H_1 be an open attached hemisphere and ξ_1 be such that the point $\eta_1\in S$ with $d_x(\xi_1,\eta_1)=\pi$ does not lie in E. Choose $\varepsilon>0$ such that $d_x(\eta_1,E)>3\varepsilon$. Then the ball $B_x(\xi_1,3\varepsilon)$ of radius 3ε centered at ξ_1 is contained in H_1. By Lemma 3.3, there is $\varphi_1\in\Gamma$ such that

$$d_x(\dot\sigma_1(0),\xi_1)<\varepsilon \text{ and } d_y\left(\varphi_1\eta_1,-\dot\sigma_1(r_1)\right)<\varepsilon\,,$$

where $y=\varphi_1x$ and $\sigma_1:[0,r_1]\to X$ is the unit speed geodesic from x to y.

Let ξ_2 be the antipode of $-\dot\sigma(r_1)$ in the central sphere φ_1S of S_yX. Then $d_y(\xi_2,\varphi_1E)>2\varepsilon$. Let H_2 be an open hemisphere attached to φ_1S and let $\eta_2\in H_2$ be a point at distance π from ξ_2. Then $B_y(\eta_2,2\varepsilon)\subset H_2$. By Lemma 3.3, there is $\varphi_2\in\Gamma$ such that

$$d_y(\dot\sigma_2(0),\xi_2)<\varepsilon \text{ and } d_z\left(\varphi_2\eta_2,-\dot\sigma_2(r_2)\right)<\varepsilon\,,$$

where $z = \varphi_2 y$ and $\sigma_2 : [0, r_2] \to X$ is the unit speed geodesic from y to z. Let $\varphi = \varphi_2\varphi_1$ and $\sigma : [0, r] \to X$ be the unit speed geodesic from x to z, see Figure 2. Since $d_y(-\dot\sigma_1(r_1), \dot\sigma_2(0)) > \pi - \varepsilon$ and since X has nonpositive curvature,

$$d_x(\dot\sigma(0), \dot\sigma_1(0)) < \varepsilon \quad \text{and} \quad d_z(-\dot\sigma(r), -\dot\sigma_2(r_2)) < \varepsilon$$

and hence $\dot\sigma(0) \in H_1$ and $-\dot\sigma(r) \in \varphi_2 H_2$.

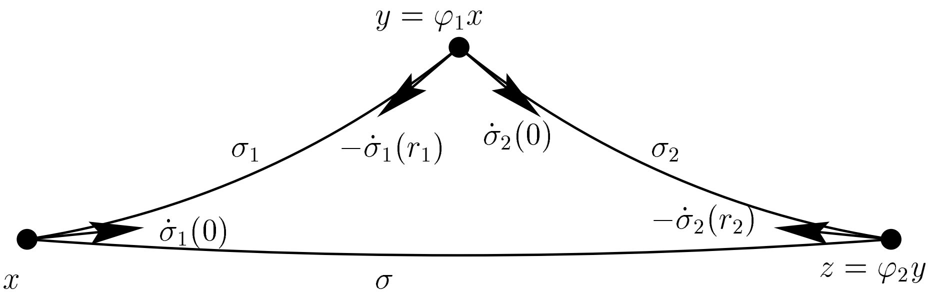

FIGURE 2

Now let H_1' be an open attached hemisphere in S_xX such that $\overline{H}_1 \cap \overline{H}_1'$ consists of two points. By Lemma 3.5, there is a direction $\xi' \in H_1'$ at distance π from $\dot\sigma(0)$ such that there are exactly two geodesics of length π from ξ' to $\dot\sigma(0)$ (their union being a geodesic of length 2π passing through $\overline{H}_1 \cap \overline{H}_1'$). Let $\widetilde\sigma : [-\varepsilon, r] \to X$ be a geodesic extension of σ with incoming direction ξ' at $x = \sigma(0) = \widetilde\sigma(0)$. If $\widetilde\sigma$ is a side of a flat rectangle R, then R is represented in the link S_xX by one of the two geodesics of length π from ξ' to $\dot\sigma(0)$. Hence (up to width) there are at most two such rectangles. Furthermore, R determines a geodesic γ of length $\pi/2$ in S_zX starting from $-\dot\sigma(r) \in \varphi_2 H_2$. We now choose another hemisphere H_2' in S_zX and a direction $\eta' \in H_2'$ of distance π to $-\dot\sigma(r)$ such that there are precisely two geodesics of length π from $-\dot\sigma(r)$ to η', neither of them containing the above geodesic γ. Hence, if $\widehat\sigma : [-\varepsilon, r+\varepsilon] \to X$ is a geodesic extension of $\widetilde\sigma$ with outgoing direction η' in z, then $\widehat\sigma$ is not a side of a flat rectangle. $\square$

4. The Case of a Spherical Join with Large Equator

From now on we assume that all links in X have diameter π, but that no link in X is a hemispherex. By the classification of [BB98], every link in X is either a thick spherical building of type A_3 or B_3 or a spherical join with an equator of diameter $= \pi$ or a spherical join with an equator of diameter $> \pi$. In the first two cases the link is a spherical building. In this section we address the case when there is a point in X whose link is a spherical join with an equator of diameter $> \pi$.

4.1. PROPOSITION. *Suppose that all links of X have diameter π, but that no link in X is a hemispherex. Let $x \in X$ be such that $S_x X$ is a spherical join with equator of diameter $> \pi$. Then X is isometric to a product of a tree with a two-dimensional Euclidean polyhedron of nonpositive curvature.*

The proof is based on several lemmas.

4.2. LEMMA. *Let $\overline{x_0 x_2}$ be a geodesic segment in X, $x_1 \in \overline{x_0 x_2}$ and $y \in X \setminus \overline{x_0 x_2}$. Suppose that the geodesic triangles (x_0, x_1, y) and (x_1, x_2, y) are flat and that $\angle_{x_1}(x_0, y) + \angle_{x_1}(x_2, y) = \pi$. Then the geodesic triangle (x_0, x_2, y) is flat.*

Proof. The Euclidean comparison triangles Δ_0 and Δ_2 for (x_0, x_1, y) and (x_1, x_2, y) are isometric to them. Hence, by the assumption on the angle sum, if we paste Δ_0 and Δ_2 along the side corresponding to $\overline{x_1 y}$, we obtain a triangle Δ which is a comparison triangle for (x_0, x_2, y). Since triangles (x_0, x_1, y) and (x_1, x_2, y) are flat, Δ has the same angles at vertices x_0 and x_2 as triangle (x_0, x_2, y). Therefore (x_0, x_2, y) is flat. □

4.3. LEMMA. *Under the assumptions of Proposition 4.1, let E_x be the equator of $S_x X$. Then there is a tree $T \subset X$ such that*

(i) *T is closed, convex, geodesically complete and contains x;*
(ii) *if $z \in T$, then $S_z X$ is a spherical join whose equator E_z is isometric to E_x and whose set of poles is $S_z T$;*
(iii) *there is $\varepsilon > 0$ such that the ε-neighborhood of T is isometric to $T \times C_\varepsilon$, where C_ε is the Euclidean cone over E_x of radius ε.*

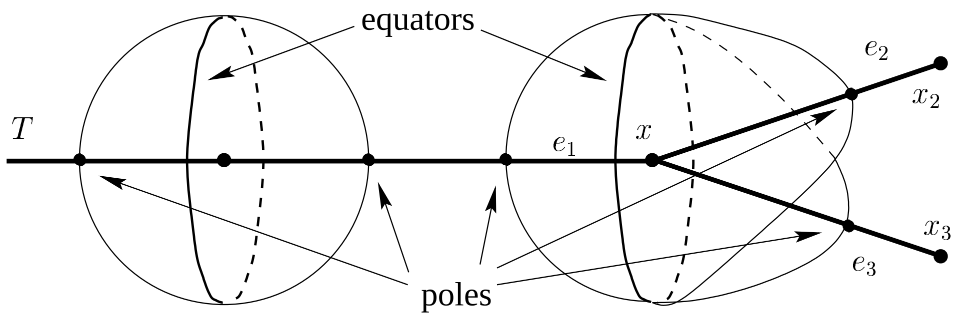

FIGURE 3

Proof. By the structure of the link S_xX, the point x is contained in the 1-skeleton of X. W.l.o.g. assume that x is a vertex. Then the poles of S_xX correspond to the directions of essential edges $e_1, \ldots, e_k$ adjacent to x, see Figure 3. Let $T_1 = \cup_i e_i$. Since X is a Euclidean polyhedron, the normal links of $e_1, \ldots, e_k$ are isometric to E_x. Let $x_1, \ldots, x_k$ be the other ends of $e_1, \ldots, e_k$, respectively. Then the link $S_{\xi_i}(S_{x_i}X)$ of the incoming direction ξ_i of e_i in $S_{x_i}X$ is also isometric to E_x. Since diam $(S_{x_i}X) = \pi$ and no link of X is a hemispherex, the link $S_{x_i}X$ is also a spherical join with equator isometric to E_x. Let T_2 be the union of T_1 with all (essential) edges corresponding to the poles of the links $S_{x_i}X$, $i = 1, \ldots, k$. Proceeding in this way we obtain a tree $T \subset X$ satisfying 1 and 2.

Since Γ acts cocompactly on X, there is $\varepsilon > 0$ such that, if a closed simplex intersects the ε-neighborhood of T, then it intersects T and 3 follows. $\square$

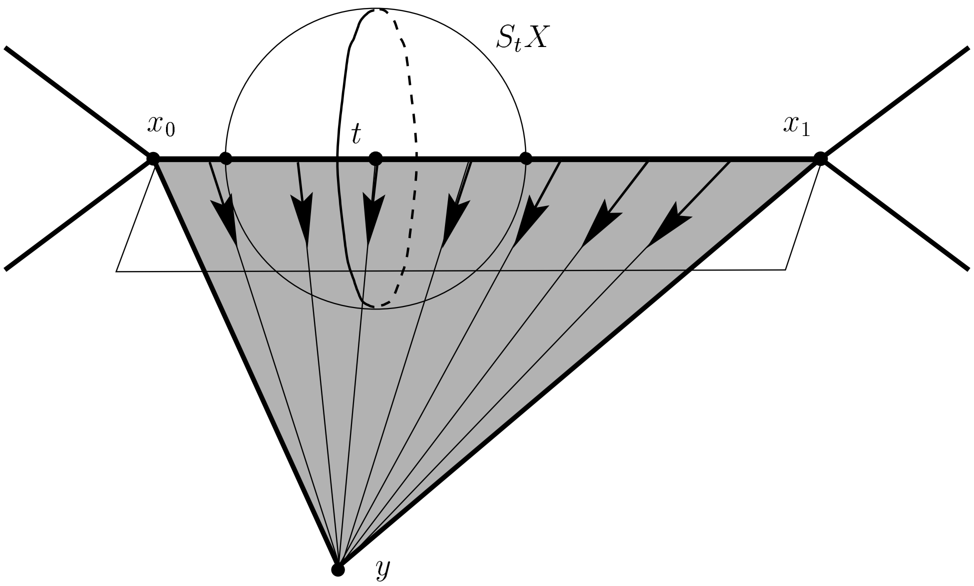

FIGURE 4

For a spherical join with equator E, away from the poles there is a well defined (nearest point) projection onto E which we call the *equatorial projection.* We say that a point $y \in X \setminus T$ is *admissible* for $z \in T$ if the equatorial projection $\xi \in E_z$ of the direction at z pointing at y has a point $\eta \in E_z$ whose distance in E_z to ξ is $> \pi$. We also say that two equatorial directions at different points of T are *parallel* along T if they are parallel in the isometric splitting $T \times C_\varepsilon$ of the ε-neighborhood of T (see Lemma 4.3) .

4.4. LEMMA. *Let $e = \overline{x_0 x_1}$ be a (closed) edge in T and let $y \in X \setminus T$ be admissible for some $z \in e$. Then triangle (x_0, x_1, y) is flat. In particular the equatorial projections of the directions pointing to y at all points of e are parallel along T.*

Proof. Let t be a point in the interior of e such that y is admissible for t, see Figure 4. Let σ_1 be the geodesic segment from t to y and let σ_2 be a geodesic segment of positive length starting at t with initial direction η whose equatorial projection lies at distance $> \pi$ to the equatorial projection of $\xi = \dot{\sigma}_1(0)$. The geodesic segment $\sigma = \sigma_1 * \sigma_2$ bounds a flat strip S which corresponds to a semicircle (of length π) $c \subset S_t X$ from ξ to η. Since the distance between the equatorial projections of ξ and η is $> \pi$, the semicircle c passes through a pole of $S_t X$. Hence S is tangent to e at t and there is a flat triangle (t, t', y) with $t' \in e$.

Let $f \subset e$ be a closed interval such that y is admissible for any $t \in f$. By the argument in the preceding paragraph, any point from the interior of f belongs to a nondegenerate flat triangle with base in f and vertex y. Note that the link of each interior point t of e is a spherical join with two poles e^+ and e^- in the directions of e. Therefore $\angle_t(\xi, e^+) + \angle_t(\xi, e^-) = \pi$ for any point t from the interior of e and any direction $\xi \in S_t X$. Hence, by Lemma 4.2, if two such flat triangles intersect, their union is a flat triangle. It follows that the interior of f is covered by a disjoint union of closed base lines of maximal, flat, nondegenerate triangles with vertex at y. Since f is an interval, we conclude that there is only one base line and the convex hull of f and y is a flat triangle.

If Δ is a flat triangle with base line in e and vertex at y, then the equatorial projections of the directions pointing to y at points on the base line are parallel along T. By what we just proved, the set of points $t \in e$ for which y is admissible, is open and closed. Therefore the convex hull of e and y is a flat triangle. $\square$

A *rectangular, flat half strip* in X is a flat triangle with two right angles and one vertex at infinity.

4.5. LEMMA. *Suppose that $y \in X \setminus T$ is admissible for some $t \in T$. Let Py be the projection of y to T and ξ be the equatorial projection of the direction pointing to y at t. Then*

(i) *y is admissible for each $t' \in T$ and the equatorial projection of the direction pointing to y at t' is parallel to ξ along T;*
(ii) *for any ray $\rho \subset T$ starting from Py, the convex hull of ρ and y is a rectangular, flat half strip $S(\rho)$ bounded by ρ, the ray $\rho(y)$ starting from y and asymptotic to ρ and the segment from y to Py.*

Proof. Let $e \subset T$ be an edge containing t. By Lemma 4.4, the convex hull of e and y is a flat triangle. It follows that the equatorial projections pointing to y at points of e are parallel along T and hence y is admissible for the ends of e. Proceeding by induction we obtain the same statement for each edge of T.

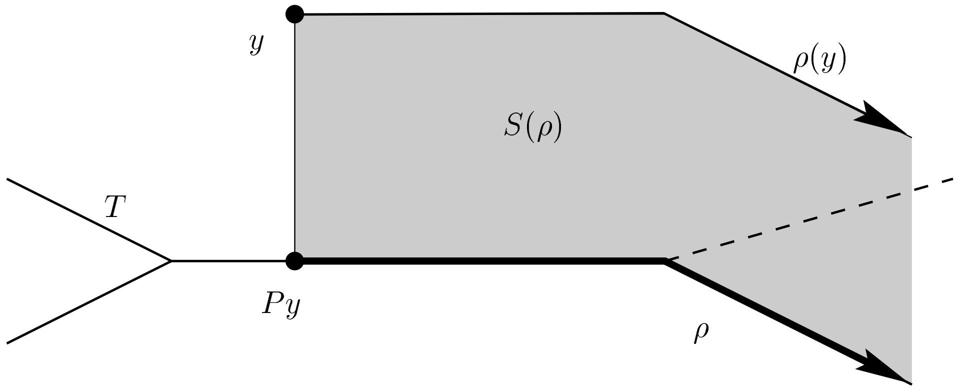

FIGURE 5

Suppose ρ is a ray in T with $\rho(0) = Py := t_0$. Let $t_1, t_2, \dots$ be consecutive vertices along ρ so that the triangles $\Delta t_n t_{n+1} y$, $n = 0, 1, \dots,$ are flat. Since ρ is a ray in T and $\angle_{t_n}(t_{n-1}, y) < \pi/2$ for $n \geq 1$, we have, by the structure of the links in T, that $\angle_{t_n}(t_{n-1}, y) + \angle_{t_n}(y, t_{n+1}) = \pi$. Hence the union $\cup_{n=0}^{\infty} \Delta t_n y t_{n+1}$ is a flat half strip, see Figure 5. $\square$

Two closed convex subsets $C, \widetilde{C} \subset X$ are *parallel* if their convex hull is isometric to the product $C \times [0, a]$, where $C \cong C \times \{0\}$, $\widetilde{C} \cong C \times \{a\}$ and $a = d(C, \widetilde{C})$.

4.6. LEMMA. *Let y be admissible for the tree T constructed in Lemma 4.3. Then there is a tree $\widetilde{T} \ni y$ which is parallel to T.*

Proof. Let $T = \cup_{n \geq 0} T_n$, where T_n is obtained from T_{n-1} by adding a ray ρ_n in T so that $T_0 = Py := x_0$ and $\rho_n \cap T_{n-1}$ consists of one point $x_n = \rho_n(0)$ for $n \geq 0$. By induction we assume that there is a tree $\widetilde{T}_{n-1} \ni y$ parallel to T_{n-1} so that the convex hull C_{n-1} of T_{n-1} and $\widetilde{T}_{n-1}$ splits as $T_{n-1} \times [0, a]$, where $a = d(y, Py)$. Let $y_n \in \widetilde{T}_{n-1}$ be the point with $Py_n = x_n$ which corresponds to (x_n, a) in the splitting of C_{n-1}. By Lemma 4.5, y_n is admissible and there is a half strip $S(\rho_n) \cong \rho_n \times [0, a]$ spanned by y_n and ρ_n. Let $\widetilde{\rho}_n = \rho_n \times \{a\}$ be the ray in $S(\rho_n)$ starting from y_n (and asymptotic to ρ_n) and set $\widetilde{T}_n = \widetilde{T}_{n-1} \cup \widetilde{\rho}_n$. Note that, by the product structure of C_{n-1} and $S(\rho_n)$, distances in $\widetilde{T}_n$ do not exceed the distances between the corresponding points in T_n.

On the other hand, for any $t_1, t_2 \in T_n$, the geodesic segments $t_1 \times [0, a]$ and $t_2 \times [0, a]$ are perpendicular to T_n. Therefore the distance between their ends is at least $d(t_1, t_2)$. Hence $\widetilde{T_n}$ is a tree parallel to T_n. □

We now finish the proof of Proposition 4.1. Let T be the tree constructed in Lemma 4.3. The closed and convex subset $Y \subset X$, consisting of points y for which there is a tree $\widetilde{T}(y) \ni y$ parallel to T, is isometric to a product $Z \times T$, where $Z \subset Y$ is closed and convex. We will show that $Y = X$.

Fix $t \in T$. For $\xi \in S_t X$ and $\alpha > 0$, denote by $C(\xi, \alpha)$ the set of points $y \in X$ such that the distance in $S_t X$ between ξ and the incoming direction of the geodesic connecting y and t is $< \alpha$. Let ξ be admissible. There is $\alpha > 0$ such that any $\eta \in S_t X$ is admissible if $\angle(\xi, \eta) < \alpha$. Hence any point $y \in C(\xi, \alpha)$ is admissible and, by Lemma 4.6, $C(\xi, \alpha) \subset Y$. Since Γ acts cocompactly, for any $R > 0$, there is $\gamma \in \Gamma$ such that $B_R(\gamma t) \subset C(\xi, \alpha)$. Therefore $B_R(t) \subset Y$. Since R is arbitrary, $Y = X$. □

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