# On dynamical coherence

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**Abstract**

A partially hyperbolic diffeomorphism is dynamically coherent if its center, center-stable, and center-unstable invariant distributions are integrable, i.e., tangent to foliations. Dynamical coherence is a key assumption in the theory of stable ergodicity. The main result: a partially hyperbolic diffeomorphism $f\colon M \to M$ is dynamically coherent if the strong stable and unstable foliations are quasi-isometric in the universal cover $\widetilde{M}$, i.e., for any two points in the same leaf, the distance between them in $\widetilde{M}$ is bounded from below by a linear function of the distance along the leaf.

Let $M$ be an $m$-dimensional differentiable manifold. A $k$-dimensional distribution $E$ on $M$ is a continuous (in $x \in M$) family of $k$-planes $E(x) \subset T_xM$.

By a $k$-dimensional *foliation* $W$ of $M$ we mean a partition of $M$ into $k$-dimensional, complete, connected, $C^1$-submanifolds $W(x) \ni x$ (called *leaves*) which (as $C^1$-submanifolds) depend continuously (in the compact – open topology) on $x \in M$. Let $D^n$ denote the open unit ball in $\mathbf{R}^n$. For each point $x \in M$, there is a *coordinate chart* $(U, \phi)$ of $W$ at $x$, i.e., a neighborhood $U$ and a homeomorphism $\phi : D^k \times D^{m-k} \to U$ such that for each $p \in D^{m-k}$, the set $W_U(\phi(0,p)) := \phi(D^k, p)$ (called the *local leaf*) is contained in $W(\phi(0,p))$ and $\phi(\cdot, p) : D^k \to W(\phi(p))$ is a $C^1$ diffeomorphism which depends continuously on $p \in D^{m-k}$ in the $C^1$-topology.

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If $W$ is a $k$-dimensional foliation on $M$, then the family of tangent planes $\{E(x) = T_xW(x)\}_{x\in M}$ is a $k$-dimensional distribution on $M$. A distribution $E$ is called *integrable* if there is a foliation $W$ whose leaves are tangent to $E$, i.e., $E(x) = T_xW(x)$ for each $x \in M$. It is not clear (to the author) to what extent the existence of such an integral foliation (globally or locally) implies its uniqueness.

If $W$ is a foliation, denote by $d_W$ the distance along the leaves of $W$ and by $W_\delta(x)$ the ball of radius $\delta$ in $W(x)$ centered at $x$.

A diffeomorphism $f$ of a compact Riemannian manifold $M$ is called *partially hyperbolic* if there are constants $0 < \lambda_1 \le \lambda_2 < \gamma_1 \le 1 \le \gamma_2 < \mu_2 \le \mu_1$, $C_{\text{ph}} \ge 1$ and, for every $x \in M$, subspaces $E^s(x)$, $E^c(x)$, $E^u(x)$ (called *stable, center* and *unstable*, respectively) such that for every $x \in M$

1. $T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$;
2. the distributions $E^s$, $E^c$, and $E^u$ are invariant under the derivative $df$, i.e., $df(x)E^\nu(x) = E^\nu(f(x))$, $\nu = u, c, s$;
3. $C_{\text{ph}}^{-1}\lambda_1^n\|v^s\| \le \|df^n(x)v^s\| \le C_{\text{ph}}\lambda_2^n\|v^s\|$ for each $v^s \in E^s(x)$ and $n > 0$;
4. $C_{\text{ph}}^{-1}\gamma_1^n\|v^c\| \le \|df^n(x)v^c\| \le C_{\text{ph}}\gamma_2^n\|v^c\|$ for each $v^c \in E^c(x)$ and $n > 0$;
5. $C_{\text{ph}}^{-1}\mu_1^n\|v^u\| \le \|df^n(x)v^u\| \le C_{\text{ph}}\mu_2^n\|v^u\|$ for each $v^u \in E^u(x)$ and $n > 0$.

It is assumed that at least one of the two distributions $E^s$ or $E^u$ is nontrivial. The stable $E^s$ and unstable $E^u$ distributions are known to be Hölder continuous [BP01] and integrable, i.e., there exist foliations $W^s$ and $W^u$, called *(strong) stable and unstable*, respectively, such that $T_xW^s(x) = E^s(x)$ and $T_xW^u(x) = E^u(x)$ for each $x \in M$ [BP74].

The center distribution $E^c$ is not always integrable [Wil98]. Its integrability was established by M. Hirsch, C. Pugh and M. Shub for small perturbations of partially hyperbolic diffeomorphisms whose center distribution is integrable and either is Lipschitz continuous or its integral foliation has compact leaves [HPS77].

A partially hyperbolic diffeomorphism is called *dynamically coherent* if the distributions $E^c$, $E^{cs}$, and $E^{cu}$ are integrable. The corresponding foliations are called the *center, center-stable*, and *center-unstable* foliations, respectively [BPSW01].

Dynamical coherence is a key assumption in the theory of stable ergodicity for partially hyperbolic diffeomorphisms which was created by C. Pugh and

M. Shub [GPS94], [PS97], [PS00] and further developed by K. Burns and A. Wilkinson [BPSW01].

The following property of a distribution implies its "unique" integrability and is similar to the uniqueness of solutions of ordinary differential equations.

A continuous $k$-dimensional distribution $E$ on a smooth Riemannian manifold $M$ is called *locally uniquely integrable* if for each $x \in M$ there are a $k$-dimensional $C^1$-submanifold $W_{\mathrm{loc}}(x)$ and $\alpha(x) > 0$ such that every piecewise $C^1$-curve $\sigma : [0,1] \to M$ satisfying (i) $\sigma(0) = x$, (ii) $\dot\sigma(t) \in E(\sigma(t))$ for $t \in [0,1]$, and (iii) length$(\sigma) < \alpha(x)$, is contained in $W_{\mathrm{loc}}(x)$. Obviously, if $E$ is locally uniquely integrable, then it is integrable and the integral foliation is unique.

If $E$ is a smooth distribution, its unique local integrability can be established by checking the Frobenius condition – the Lie brackets of vector fields tangent to $E$ must be tangent to $E$.

The main result of this paper is the integrability of the center distribution (and its joint integrability with the stable and unstable distributions) for partially hyperbolic diffeomorphisms whose stable and unstable foliations are quasi-isometric.

A foliation $W$ of a simply connected Riemannian manifold $\widetilde{M}$ is called *quasi-isometric* [Fen92] if there are positive constants $a$ and $b$ such that for any two points $x$ and $y$ which lie in the same leaf of $W$,

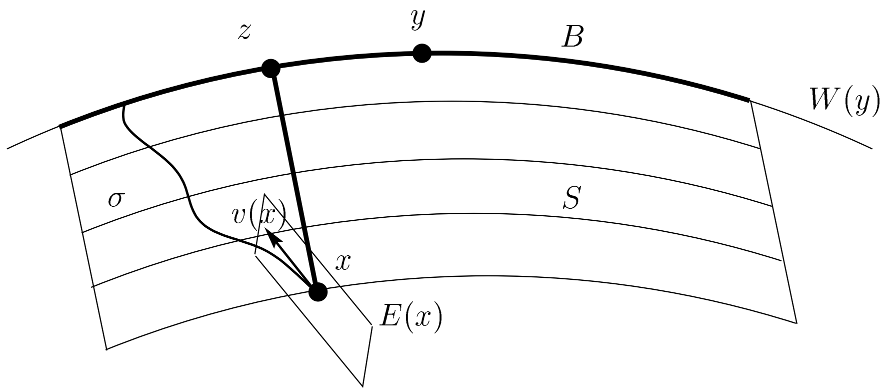
$$d_W(x,y) \leq a \cdot d(x,y) + b\,.$$

**THEOREM 1.** *Let $f$ be a partially hyperbolic diffeomorphism of a compact $m$-dimensional Riemannian manifold $M$. Suppose the stable $W^s$ and unstable $W^u$ foliations of $f$ are quasi-isometric in the universal cover $\widetilde{M}$.*

*Then the distributions $E^c$, $E^{cs} = E^c \oplus E^s$, and $E^{cu} = E^c \oplus E^u$ are locally uniquely integrable; in particular, $f$ is dynamically coherent.*

*Proof.* We will prove the local unique integrability of $E^{cs}$. The local unique integrability of $E^{cu}$ and the existence of the center-unstable foliation $W^{cu}$ follow by reversing the time. The local unique integrability of $E^c$ and the existence of the center foliation $W^c$ are established by observing that every $C^1$-curve starting at $x \in M$ and tangent to $E^c$ is also tangent to $E^{cs}$ and to $E^{cu}$ and therefore is contained in $W^{cs}(x) \bigcap W^{cu}(x)$.

**LEMMA 2.** *Suppose $W$ is a $k$-dimensional foliation of $M$ and $E$ is an $l$-dimensional distribution such that $k+l = m$ and $E$ is (uniformly) transverse to $W$, i.e., $T_xW(x) \cap E(x) = \{0\}$ for each $x \in M$.*




*There is $C_0 > 0$ and for every $\delta > 0$ there is $\epsilon_0 > 0$ such that if $d(x, y) < \epsilon_0$ then there is a $C^1$-curve $\sigma : [0, 1] \to M$ with the property that $\sigma(0) = x$, $\sigma(1) \in W_\delta(y)$, $\dot\sigma(t) \in E(\sigma(t))$ for $t \in [0, 1]$, length$(\sigma) \le C_0 d(x, y)$, and $d_W(y, \sigma(1)) \le C_0 d(x, y)$.*

*Proof.* By choosing a small enough $\epsilon_0$ we may assume that a neighborhood of $x$ under consideration is identified by a coordinate chart (with uniformly bounded derivative) with the unit ball in $\mathbf{R}^m$. For a small enough $\delta > 0$, the ball $B = W_\delta(y)$ is contained in a local leaf and is identified with a $k$-dimensional $C^1$-submanifold in $\mathbf{R}^m$. Assuming that $\epsilon_0$ is small enough, there is a point $z \in B$ for which $d(x, z) = d(x, B)$ and $d_W(z, y) \le Cd(x, y)$, where $C$ depends only on $W$. In particular $d(x, z) \le d(x, y)$ and $x - z \perp T_z B$. The union

$$S = \bigcup_{0 \le t \le 1} (B + (1 - t)(x - z))$$

of the parallel translates of the local leaf $B$ along the segment connecting $x$ to $y$ is a $(k + 1)$-dimensional $C^1$-surface which is a Riemannian product of the interval $[0, \|x - z\|]$ and $B$. We refer to $B$ and $x - z$ as the horizontal and vertical components, respectively. For $s \in S$, let $L(s) = E(s) \bigcap T_s S$. Since $E$ is uniformly transverse to $B$, if $\delta$ and $\epsilon_0$ are small enough, then $L$ is a continuous line field $L$ on $S$ which is uniformly (in $x$, $y$, and $s$) transverse to the codimension 1 (in $S$) submanifolds $B + (1 - t)(x - z)$. Therefore, for each $s \in S$ there is a unique vector $v(s) \in L(s)$ whose vertical component is $z - x$. The uniform transversality of $E$ and $B + (1 - t)(x - z)$ implies that $v$ is a uniformly (in $x$, $y$, and $s$) bounded, continuous vector field on $S$. For each

$s \in S$, the vertical component of $v(s)$ is $z - x$ and the horizontal component is bounded by a constant $C'$ which depends on $\epsilon_0$ and $\delta$ but not on $x$, $y$, and $s$. By the existence theorem for ordinary differential equations, there is a (possibly not unique) solution $\sigma(\cdot)$ of $\dot{s} = v(s)$ with $\sigma(0) = x$. The solution stays within the $C'$-cone near the vertical segment in $S$. Therefore, if $\epsilon_0$ and $\delta$ are small enough, its interval of definition can be extended to $[0, 1]$ so that $\sigma(1) \in B$, $\text{length}(\sigma) \le \sqrt{1 + C'^2} d(x, z) \le C_0 d(x, y)$, and $d_W(y, \sigma(1)) \le C_0 d(x, y)$. $\square$

To prove the unique integrability of $E^{cs}$, lift all distributions and foliations to the universal cover $\widetilde{M}$. Fix $r \in (0, \epsilon_0)$. For $x \in \widetilde{M}$ denote by $W^{cs}_r(x)$ the set of ends of all piecewise $C^1$-curves $\sigma : [0, 1] \to M$ such that $\sigma(0) = x$, $\dot{\sigma}(t) \in E^{cs}(\sigma(t))$, and $\text{length}(\sigma) < r$.

**LEMMA 3.** *Let $y, z \in \widetilde{M}$ be such that $y, z \in W^{cs}_r(x)$ and $z \in W^u(y)$. Then $z = y$.*

*Proof.* By assumption, there are $C^1$-curves $\sigma_y$ and $\sigma_z$ tangent to $E^{cs}$ and connecting $x$ to $y$ and $z$, respectively. Consider the points $f^n(x)$, $f^n(y)$, $f^n(z)$. From the partial hyperbolicity inequalities we have:

$$\text{length}(f^n(\sigma_y)) \le C_{\text{ph}} \gamma_2^n \cdot \text{length}(\sigma_y), \quad \text{length}(f^n(\sigma_z)) \le C_{\text{ph}} \gamma_2^n \cdot \text{length}(\sigma_z)$$

and therefore

$$d(f^n(y), f^n(z)) \le C_{\text{ph}} \gamma_2^n (\text{length}(\sigma_y) + \text{length}(\sigma_z)) .$$

On the other hand, let $d_u$ denote the distance along the leaves of $W^u$ and let $\sigma_n$ be a $C^1$-curve in $W^u(f^n(y)) = W^u(f^n(z))$ which realizes $d_u(f^n(y), f^n(z))$. Then

$$d_u(f^n(y), f^n(z)) = \text{length}(\sigma_n) \ge C_{\text{ph}}^{-1} \mu_2^n \text{length}(f^{-n}(\sigma_n)) \ge C_{\text{ph}}^{-1} \mu_2^n d_u(y, z) .$$

Since $W^u$ is a quasi-isometric foliation,

$$d(f^n(y), f^n(z)) \ge (d_u(f^n(y), f^n(z)) - b)/a \ge \left(C_{\text{ph}}^{-1} \mu_2^n d_u(y, z) - b\right)/a .$$

Since $d(f^n(x), f^n(y)) \le \text{length}(f^n(\sigma_y))$, $d(f^n(x), f^n(z)) \le \text{length}(f^n(\sigma_z))$, and since $\gamma_2 < \mu_2$, we obtain a contradiction with the triangle inequality for $f^n(x)$, $f^n(y)$, $f^n(z)$ (with $n$ sufficiently large) unless $d_u(y, z) = 0$. $\square$

We need to show that $W_r^{cs}(x)$ is a $C^1$-submanifold of the same dimension as $E^{cs}$.

Fix $\delta > 0$ such that for each $x \in \widetilde{M}$, the $\delta$-ball $W_\delta^u(x)$ is contained in a local leaf of $W^u$. For $\epsilon > 0$ and $y \in W_r^{cs}(x)$, denote by $E_\epsilon^{cs}(y)$ the $\epsilon$-ball in $E^{cs}(y)$ centered at $y$. Since $W^u(y)$ is transverse to $E^{cs}(y)$ at $y$, for small enough $0 < \epsilon \ll \delta$ and each $z \in E_\epsilon^{cs}(y)$, the leaf $W^u(z)$ is transverse to $E^{cs}(y)$ at $z$. By Lemma 2, if $r$ and $\epsilon < r$ are small enough, then each leaf $W_\delta^u(z)$ intersects $W_r^{cs}(x)$ and, by Lemma 3, the intersection consists of a single point $\pi(z)$. Let $z_1, z_2 \in E_\epsilon^{cs}(y)$. Since $W_\delta^u(z)$ depends continuously on $z$ as a $C^1$-submanifold, $d(\pi(z_1), W_\delta^u(z_2)) \to 0$ as $d(z_1, z_2) \to 0$. Therefore, by Lemma 2, $d(\pi(z_1), \pi(z_2) \to 0$ as $d(z_1, z_2) \to 0$. It follows that $\pi$ is continuous and a homeomorphism onto its image which is a neighborhood of $y = \pi(y)$ in $W_r^{cs}(x)$. It follows that $W_r^{cs}(x)$ is a topological submanifold. Obviously $E^{cs}(y)$ is the tangent plane to $W_\epsilon^{cs}(y)$ at $y$. Since the distribution $E^{cs}$ is continuous, $W_r^{cs}(x)$ is a $C^1$ submanifold. Hence $E^{cs}$ is locally uniquely integrable and the leaf $W^{cs}(x)$ of the integral foliation $W^{cs}$ passing through $x \in M$ is the set of ends of piecewise $C^1$-curves starting at $x$ and tangent to $E^{cs}$. This completes the proof of Theorem 1. □

In general, the stable and unstable foliations of a partially hyperbolic diffeomorphism do not have to be quasi-isometric. For example, for the time-1 map of the geodesic flow on a compact manifold of negative sectional curvature, the stable and unstable foliations are not quasi-isometric – the unstable horosphere expands exponentially with time but the distance between points on the horosphere grows linearly in time.

**PROPOSITION 4.** *Let $W$ be a $k$-dimensional foliation of the $m$-dimensional torus $\mathbf{T}^m$. Suppose there is an $(m-k)$-dimensional plane $A$ such that $T_xW(x) \bigcap A = \emptyset$ for each $x \in \mathbf{T}^m$.*

*Then the lift $\widetilde{W}$ (of $W$ to $\mathbf{R}^m$) is quasi-isometric.*

*Proof.* Let $\widetilde{B}$ be the orthogonal complement of the lift $\tilde{A}$ of $A$ in $\mathbf{R}^m$. Fix $x \in \mathbf{R}^m$ and let $\pi : \widetilde{W}(x) \to \widetilde{B}$ be the orthogonal (parallel to $\tilde{A}$) projection of $\widetilde{W}(x)$ to $\widetilde{B}$. By compactness, $\widetilde{W}$ is uniformly transverse to $\tilde{A}$. Therefore there is $\alpha > 0$ such that $\|d\pi(y)v\| \geq \alpha\|v\|$ for each $y \in \widetilde{W}(x)$ and $v \in T_y\widetilde{W}(x)$. Hence, by compactness, there is $\delta > 0$ such that for each $y \in \widetilde{W}(x)$, the map $\pi$ restricted to the $\delta$-neighborhood $\widetilde{W}_\delta(y)$ of $y$ in $\widetilde{W}(x)$ is a diffeomorphism onto its image which contains the $\alpha\delta$-neighborhood of $\pi(y)$. It follows that $\pi$ is a covering map. Since $\widetilde{W}(x)$ and $\widetilde{B}$ are simply connected, $\pi$ is one-to-one.

Let $y, z \in \widetilde{W}(x)$ and let $\sigma : [0,1] \to \widetilde{B}$ be the straight line segment connecting $\pi(y)$ to $\pi(z)$. The lift $\pi^{-1} \circ \sigma$ is a $C^1$-curve connecting $y$ to $z$. By uniform transversality,

$$d(\pi(y), \pi(z)) = \text{length}(\sigma) \geq \frac{1}{\alpha}\text{length}(\pi^{-1} \circ \sigma) \geq \frac{1}{\alpha} d_{\widetilde{W}}(y, z),$$

where, as before, $d_{\widetilde{W}}(y, z)$ is the distance from $y$ to $z$ in $\widetilde{W}(x)$. $\square$

The next proposition is an application of Proposition 4 to a certain class of partially hyperbolic diffeomorphisms of $\mathbf{T}^m$.

**PROPOSITION 5.** *Let* $f_0 : \mathbf{T}^3 \to \mathbf{T}^3$ *act by*

$$f_0(x, y, z) = \left(A\begin{pmatrix} x \\ y \end{pmatrix}, z + \alpha\right) \bmod 1,$$

*where* $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ *and* $\alpha \in \mathbf{R}$. *Suppose* $f_1 = g \circ f_0$, *where* $g : \mathbf{T}^3 \to \mathbf{T}^3$ *is a* $C^1$*-diffeomorphism such that* $\|dg(p) - \mathbf{I}\| < 0.1$ *for each* $p \in \mathbf{T}^3$. *Then* $f_1$ *is partially hyperbolic and its weak stable, weak unstable, and center distributions are locally uniquely integrable.*

*Proof.* Use a subindex 0 for the invariant distributions of $f_0$ and 1 for those of $f_1$. The stable $E_0^u(x)$, unstable $E_0^s(x)$, and center $E_0^c(x)$ subspaces for $f_0$ are 1-dimensional and mutually perpendicular. The subspaces at different points are also parallel, so we omit the dependence on $x \in \mathbf{T}^3$. The derivative $df_0$ stretches vectors from $E_0^u$ by a factor of $\dfrac{3+\sqrt{5}}{2} \approx 2.6180$, contracts vectors from $E_0^s$ by a factor of $\dfrac{3-\sqrt{5}}{2} \approx .3820$, and acts as an isometry in the direction of $E_0^c$. For a tangent vector $v$, let $v = v^u + v^s + v^c$ be the decomposition given by the splitting $E_0^u \oplus E_0^s \oplus E_0^c$.

Consider the strong and weak, stable and unstable $\pi/6$-cones

$$K_0^u = \{v : \|v^s + v^c\| \leq .5\}, \qquad K_0^{cu} = \{v : \|v^s\| \leq .5\},$$
$$K_0^s = \{v : \|v^u + v^c\| \leq .5\}, \qquad K_0^{cs} = \{v : \|v^u\| \leq .5\}.$$

A direct computation shows that the strong and weak unstable cones are invariant under $df_1$ and the strong and weak stable cones are invariant under

$df_1^{-1}$. It follows that $f_1$ has invariant distributions $E_1^u$, $E_1^s$, $E_1^{cu}$, $E_1^{cs}$ which lie in the $\pi/6$-cones of the corresponding distributions of $f_0$.

Again, a direct computation shows that each vector from $K_0^u$ is stretched by $df_1$ by a factor of at least 2 and each vector from $K_0^s$ is stretched by $df_1^{-1}$ by a factor of at least 2.

Let $v \in E_1^c$. Then $\|v^s\| \le .5\|v\|$ and $\|v^u\| \le .5\|v\|$. A direct computation shows that $1.8^{-1}\|v\| \le \|df_1 v\| \le 1.8\|v\|$. Therefore $f_1$ is partially hyperbolic.

The stable $W_1^s$ and unstable $W_1^u$ foliations of $f_1$ satisfy the assumptions of Proposition 4 with $A = E_0^{cu}$ for $W_1^s$ and $A = E_0^{cs}$ for $W_1^u$ and the proposition follows from Theorem 1. □

A partially hyperbolic diffeomorphism $f : M \to M$ is *center-isometric* if it acts isometrically in the center direction, i.e., $\|df(x)v\| = \|v\|$ for every $x \in M$ and $v \in E^c(x)$.

**PROPOSITION 6.** *If a partially hyperbolic diffeomorphism $f : M \to M$ is center-isometric, then the distributions $E^c$, $E^{cs}$, and $E^{cu}$ are locally uniquely integrable; in particular, $f$ is dynamically coherent.*

*Proof.* The proof of the proposition is completely analogous to the proof of Theorem 1 with the only exception that Lemma 3 should be replaced by the following lemma. □

**LEMMA 7.** *There are $r_0 > 0$ and $\delta_0 > 0$ such that for every $r \in (0, r_0]$ and $\delta \in (0, \delta_0]$ if $y, z \in M$, $y, z \in W_r^{cs}(x)$, and $z \in W_\delta^u(y)$, then $z = y$.*

*Proof.* Since the leaves of $W^u$ are $C^1$ and depend continuously on the point, there is $\delta_0 > 0$ and $C_1 > 0$ such that if $z' \in W_{\delta_0}^u(y')$, then

$$d_u(y', z') \le C_1 d(y', z').$$

By the uniform expansion of $W^u$,

$$d_u(f^n(y), f^n(z)) \ge C_{\mathrm{ph}}^{-1} \mu_2^n d_u(y, z).$$

Assuming that $z \ne y$, choose $n > 0$ so that $d_u(f^k(y), f^k(z)) < \delta_0$ for $k = 0, 1, \ldots, n$ and $d_u(f^{n+1}(y), f^{n+1}(z)) \ge \delta_0$. Then

$$d_u(f^n(y), f^n(z)) \ge \nu\delta_1,$$

where $\nu$ is the maximum of the norm of $df$ over $M$.

By assumption, there are $C^1$-curves $\sigma_y$ and $\sigma_z$ which have length $\leq r$, are tangent to $E^{cs}$ and connect $x$ to $y$ and $z$, respectively. Consider the points $f^n(x)$, $f^n(y)$, $f^n(z)$. By the partial hyperbolicity and center isometry,

$$\text{length}(f^n(\sigma_y)) \leq C_{\text{ph}} \cdot \text{length}(\sigma_y), \quad \text{length}(f^n(\sigma_z)) \leq C_{\text{ph}} \cdot \text{length}(\sigma_z)$$

and therefore

$$d(f^n(y), f^n(z)) \leq C_{\text{ph}}(\text{length}(\sigma_y) + \text{length}(\sigma_z)) \leq 2C_{\text{ph}} r\,.$$

It follows that

$$C_1^{-1}\delta_0 \leq d(f^n(y), f^n(z)) \leq 2C_{\text{ph}} r$$

which gives a contradiction for a small enough $r$. □

Let $f : \mathbf{T}^3 \to \mathbf{T}^3$ be a partially hyperbolic diffeomorphism (with one-dimensional stable, center, and unstable distributions). The following questions arise naturally:

1) Are the stable and unstable foliations necessarily quasi-isometric?

2) Is $f$ necessarily dynamically coherent?

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