ORBIHEDRA OF NONPOSITIVE CURVATURE

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Abstract. A 2-dimensional orbihedron of nonpositive curvature is a pair (X, Γ), where X is a 2-dimensional simplicial complex with a piecewise smooth metric such that X has nonpositive curvature in the sense of Alexandrov and Busemann and Γ is a group of isometries of X which acts properly discontinuously and cocompactly. By analogy with Riemannian manifolds of nonpositive curvature we introduce a natural notion of rank 1 for (X, Γ) which turns out to depend only on Γ and prove that, if X is boundaryless, then either (X, Γ) has rank 1, or X is the product of two trees, or X is a thick Euclidean building. In the first case the geodesic flow on X is topologically transitive and closed geodesics are dense.

1. Introduction

The idea of considering curvature bounds on metric spaces belongs to Alexandrov [Ale], Busemann [Bus] and Wald [Wal]. Busemann initiated the theory of spaces of nonpositive curvature. Later, Bruhat and Tits [BrTi] showed that there is a natural metric of nonpositive curvature on Euclidean buildings and used it to prove a generalization of the theorem of Cartan on maximal compact subgroups of semisimple Lie groups. The work of Gromov (see for example [Gr1] and [Gr2]) led to a revival of the general theory of metric spaces with curvature bounds and to applications in Riemannian geometry, combinatorial group theory and other fields.

In this paper we discuss the rank rigidity for singular spaces of nonpositive curvature. To a large extent the main concepts and ideas introduced below are a natural development of the corresponding aspects of the rank rigidity theory for Riemannian manifolds of nonpositive curvature (see [BBE],[BBS],[Ba3],[BuSp],[EbHe]).

An **orbispace** is a pair (X, Γ), where X is a simply connected topological space and Γ is a group of homeomorphisms of X acting properly discontinuously. An orbispace (X, Γ) is **compact** if Γ acts cocompactly. An orbispace (X, Γ) is an **orbihedron** if X admits a Γ-invariant triangulation.

We are interested in orbispaces and orbihedra of nonpositive curvature, that is, we require in addition that X has a complete Γ-invariant geodesic metric d of nonpositive curvature in the sense of Alexandrov and Busemann. As in the smooth case, asymptote classes of geodesic rays define the space $X(\infty)$ of points at infinity and $\overline{X} = X \cup X(\infty)$

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has a natural Γ-invariant topology. If X is locally compact, then $\overline{X}$ is compact. Here are some examples of spaces of nonpositive curvature.

1. Smooth Riemannian manifolds with nonpositive sectional curvature.
2. Trees with interior metrics.
3. Euclidean buildings with their canonical metrics (defined up to a constant), see [Bro, Chapter VI].
4. (m, n)-spaces for $mn \geq 2(m+n)$ with their canonical piecewise flat metrics.

Such a space is a 2-dimensional CW-complex X with the following properties: the attaching maps are local homeomorphisms, the boundary of each face consists of at least m edges (counted with multiplicity) and every simple loop in the link (see Section 2 for the definition of a link) of a vertex consists of at least n edges. The natural flat metric on X makes each face of X with k edges an isometrically immersed regular Euclidean k-gon. Such spaces arise naturally in combinatorial group theory, see [LySc] and [BaBr].

Let (X, Γ) be a compact orbispace of nonpositive curvature. A geodesic $\sigma : \mathbb{R} \to X$ is called Γ-**closed** if there is an isometry $\phi \in \Gamma$ translating σ that is, $\phi(\sigma(t)) = \sigma(t + t_0)$ for some $t_0 \neq 0$ and all $t \in \mathbb{R}$. A Γ-closed geodesic σ and an isometry $\phi \in \Gamma$ translating σ are said to have **rank 1** if σ does not bound a flat half plane. The orbispace (X, Γ) has **rank 1** if Γ contains a rank 1 isometry.

Theorem A. *Let (X, Γ) be a compact orbispace of rank 1 and suppose that $X(\infty)$ contains more than two points.*

Then for any two nonempty open subsets $U, V \subset X(\infty)$ there is $\phi \in \Gamma$ such that $\phi(X(\infty) \setminus U) \subset V$ and $\phi^{-1}(X(\infty) \setminus V) \subset U$. Moreover, there is a Γ-closed geodesic σ of rank 1 with $\sigma(-\infty) \in U$ and $\sigma(\infty) \in V$.

This is the key property of rank 1 orbispaces. Applications of this property are discussed in Theorems D and E below.

Theorem B. *The property of a compact orbispace (X, Γ) of nonpositive curvature to have rank 1 depends only on Γ.*

This theorem generalizes a result of Eberlein [Eb2]. The main idea of the proof goes back to Morse [Mor]. In fact, we obtain an algebraic criterion for an element of Γ to have rank 1. Theorems A and B are our main motivation for considering rank 1 orbispaces.

Let X be a locally finite simplicial complex. **A piecewise smooth Riemannian metric** g on X is a family of smooth Riemannian metrics g_A on the simplices A of X such that $g_A|_B = g_B$ for any simplices B and A with $B \subset A$.

Let g be a piecewise smooth Riemannian metric on X. Then the lengths of curves in X are defined and the induced distance function d makes X an interior metric space. Assume that there is a uniform bound on the geometry of the simplices in X. Then d is complete, and hence, geodesic, since X is locally compact.

Let (X, Γ) be an orbihedron with X locally finite. We say that a Γ-invariant metric d on X is **piecewise smooth** if there is a Γ-invariant triangulation on X with a Γ-invariant piecewise smooth Riemannian metric g such that d is the induced distance function.

A k-simplex in X is called a **boundary** simplex if it is adjacent to exactly one $(k+1)$-simplex. An n-dimensional orbihedron (X, Γ) is called **boundaryless** (or we say that it is **without boundary**) if there are no boundary simplices in X with respect to some (and hence any) triangulation of X. If (X, Γ) is a 2-dimensional orbihedron of nonpositive curvature, then X contains a Γ-invariant suborbihedron X' without boundary and of dimension ≤ 2 such that a) X' is a Γ-equivariant strong deformation retract of X, b) the action of Γ on X' is effective and c) the induced metric on X' is piecewise smooth and of nonpositive curvature (see Section 2). If (X, Γ) is a compact 2-dimensional orbihedron of nonpositive curvature, then every geodesic segment of X is contained in a complete geodesic (one defined on the whole real line) iff (X, Γ) is boundaryless.

The following theorem is the main result of this paper.

Theorem C. *Let (X, Γ) be a compact 2-dimensional boundaryless orbihedron with a piecewise smooth metric of nonpositive curvature. Then either*

(i) *(X, Γ) is of rank 1, or*
(ii) *X is the product of two trees endowed with the product metric of two interior metrics, or*
(iii) *X is a thick Euclidean building of type A_2, B_2 or G_2, endowed with its canonical metric.*

In Cases (ii) and (iii) every geodesic is contained in a flat plane. In Case (i) we prove that there is a Γ-closed geodesic σ such that either a) σ passes through a point in an open face where the Gauss curvature of X is negative, or b) σ passes from one face to another through a point in an open edge e where the sum of the geodesic curvatures of e with respect to the two faces is negative, or c) σ passes through a vertex v and the distance between the incoming and outgoing directions of σ in the link of v is $> \pi$. We call σ **hyperbolic** if one of the cases a), b), or c) occurs.

Denote by $G(X)$ the set of unit speed geodesics $\sigma : \mathbb{R} \to X$, endowed with the compact-open topology. The **geodesic flow** g^t acts on $G(X)$ by $g^t(\sigma)(s) = \sigma(s+t)$.

Theorem D. *Let (X, Γ) be a compact 2-dimensional boundaryless orbihedron with a piecewise smooth metric of nonpositive curvature. If (X, Γ) has rank 1, then*

(i) *hyperbolic Γ-closed geodesics are dense in the space of geodesics;*
(ii) *the geodesic flow is topologically transitive modulo Γ.*

Note that in Cases (ii) and (iii) of Theorem C the geodesic flow has continuous first integrals. In a later paper we will study the asymptotics of the number of Γ-closed geodesics. In the rank 1 case the arguments are very similar to the arguments of G.Knieper in [Kni].

Theorem E. *Let (X, Γ) be a compact 2-dimensional boundaryless orbihedron with a piecewise smooth metric of nonpositive curvature.*

Then either Γ contains a free nonabelian subgroup or X is isometric to the Euclidean plane and Γ is a Bieberbach group of rank 2.

This result is analogous to the Tits theorem on free subgroups of linear groups, see [Ti1]. Theorem E is a consequence of Theorem C and the following result.

Theorem F. *Let X be a Euclidean building and Γ a group of automorphisms of X acting properly discontinuously and cocompactly.*

Then either Γ *contains a free nonabelian subgroup or* X *is a Euclidean space and* Γ *is a Bieberbach group whose rank is* $\dim X$.

For thick Euclidean buildings of dimension ≥ 3, this follows from Tits' theorem quoted above [Ti1] and his classification of spherical buildings of rank ≥ 3 (see [Ti2]).

The proof of our main result, Theorem C, consists of two parts. In the first part we consider the case when all faces of X are flat Euclidean triangles and all links have diameter π and show by an elementary argument that X is either a product of two trees or a thick Euclidean building of type A_2, B_2 or G_2. This part is related to the (unpublished) result of B.Kleiner [Kle] that if every geodesic of an n-dimensional complete simply connected space X of nonpositive curvature is contained in an n-flat, then X is a Euclidean building or a product of Euclidean buildings.

In the second and main part of the argument we start by considering the following cases: a) the Gaussian curvature of a face is negative at an interior point, b) there is an edge e with adjacent faces f_1 and f_2 and a point x in the interior of e such that the sum of the geodesic curvatures of e at x with respect to f_1 and f_2 is negative. In both cases we conclude that (X, Γ) has rank 1, thus reducing the general discussion to the case when all faces of X are Euclidean triangles but at least one of the links has diameter $> \pi$. The existence of a rank 1 isometry in Γ in the latter case would follow easily if there were a Γ- and g^t-invariant measure which is positive on open sets of geodesics. In Section 3 we construct a natural generalization of the Liouville measure which is invariant under isometries and the geodesic flow. However, this measure is concentrated on the set of geodesics that do not pass through vertices.

After discussing some preliminaries in Section 2, we construct an analogue of the Liouville measure in Section 3. Theorems A and D are proved in Section 4, Theorem B in Section 5, Theorem C in Sections 6 and 7, Theorems E and F in Section 8. Sections 4 and 5, Sections 6 and 7, and Section 8 can be read independently.

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2. Preliminaries

Let X be a metric space with metric d. A curve $c : I \to X$ is called a **geodesic** if there is $v \geq 0$, called the speed, such that every $t \in I$ has a neighborhood $U \subset I$ with $d(c(t_1), c(t_2)) = v|t_1 - t_2|$ for all $t_1, t_2 \in U$. If the above equality holds for all $t_1, t_2 \in I$, then c is called **minimal** or **minimizing**.

We say that X is a **geodesic space** if every two points in X can be connected by a minimal geodesic. A locally compact and complete metric space X is geodesic if it is **interior**, that is, if the distance between every two points in X is the infimum of the lengths of curves connecting them. We assume from now on that X is a complete geodesic space.

A **triangle** Δ in X is a triple $(\sigma_1, \sigma_2, \sigma_3)$ of geodesic segments whose end points match in the usual way. Denote by S_K the simply connected complete surface of constant Gauss curvature K. A **comparison triangle** $\overline{\Delta}$ for a triangle $\Delta \subset X$ is a triangle in S_K with the same lengths of sides as Δ. A comparison triangle in S_K exists and is unique up to congruence if the lengths of the sides of Δ satisfy the triangle inequality and, in the case $K > 0$, if the perimeter of Δ is $< 2\pi/\sqrt{K}$. Let $\overline{\Delta} = (\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3})$ be a comparison triangle for $\Delta = (\sigma_1, \sigma_2, \sigma_3)$, then for every point $x \in \sigma_i$, $i = 1, 2, 3,$ we denote by $\bar{x}$ the unique point on $\overline{\sigma_i}$ which lies at the same distances to the ends as x.

Let d denote the distance functions in both X and S_K. A triangle Δ in X is a $\mathbf{CAT_K}$ **triangle** if the sides satisfy the triangle inequality, the perimeter of Δ is $< 2\pi/\sqrt{K}$ for $K > 0$, and if

$$d(x, y) \leq d(\bar{x}, \bar{y})$$

for every two points $x, y \in \Delta$.

We say that X has curvature at most K and write $K_X \leq K$ if every point $x \in X$ has a neighborhood U such that any triangle in X with vertices in U and minimizing sides is CAT_K. Note that we do not define K_X. If X is a Riemannian manifold, then $K_X \leq K$ iff K is an upper bound for the sectional curvature of X.

We call X a **Hadamard space** if X is simply connected, complete, geodesic with $K_X \leq 0$. The following result is proved in [AlBi], see also [Ba2].

2.1 Hadamard-Cartan Theorem. *If X is a Hadamard space, then*

- (i) *for any two points $x, y \in X$ there is a unique geodesic $\sigma_{xy} : [0,1] \to X$ from x to y and σ_{xy} is continuous in x, y;*
- (ii) *every triangle in X is CAT_0.*

Let X be a Hadamard space and let σ_1, σ_2 be two unit speed geodesic rays going out of $x \in X$. We define $\angle(\sigma_1, \sigma_2)$ in the following way. Let $\overline{\Delta}$ be a comparison triangle with $K = 0$ for the triangle $\Delta = (\sigma_1([0, s]), \sigma_2([0, t]), \sigma_{st})$, where σ_{st} is the geodesic from $\sigma_1(s)$ to $\sigma_2(t)$. Since triangles in X are CAT_0, the angle $\alpha(s, t)$ of $\overline{\Delta}$ at $\bar{x}$ decreases as s and t decrease. We set

$$\angle(\sigma_1, \sigma_2) = \lim_{s,t \to 0} \alpha(s, t) .$$

Let $x, y, z \in X$ with $y, z \neq x$ and let γ_{xy}, γ_{xz} be the unit speed geodesics from x to y, z respectively. Set

$$\angle_x(y,z) = \angle(\gamma_{xy}, \gamma_{xz}).$$

2.2 Proposition. *(See [Ba2], Proposition 1.5.2.) Let X be a Hadamard space and let Δ be a triangle in X with sides of length a, b, c and angles α, β, γ at the opposite vertices, respectively. Then*

(i) $\alpha + \beta + \gamma \leq \pi$;

(ii) *(Cosine Inequality)* $\quad c^2 \geq a^2 + b^2 - 2ab\cos\gamma$.

In each case, equality holds if and only if Δ is flat, that is, Δ is the boundary of a 2-dimensional convex region in X isometric to the region bounded by the comparison triangle in the flat plane. □

It follows easily that for any two geodesics σ_1, σ_2 in X, the function $d(\sigma_1(t), \sigma_2(t))$ is convex in t. A more special property is as follows.

2.3 Corollary. *Let X be a Hadamard space. Let $x_1, x_2 \in X$ be two distinct points and let σ be the geodesic connecting them. Assume that σ_i is a geodesic ray going out of x_i and making angle α_i with σ, $i = 1, 2$. Suppose that $\alpha_1 + \alpha_2 > \pi$.*

Then $d(\sigma_1(t), \sigma_2(t))$ is strictly increasing in t. □

2.4 Lemma. *Let X be a locally compact Hadamard space and $Y \subset X$ be a path connected, closed, locally convex subset. Then Y is convex.*

Proof. For every compact subset $K \subset Y$ there is $\varepsilon > 0$ such that if $x, y \in K$ and $d(x,y) < \varepsilon$ then the geodesic σ_{xy}, connecting x to y, lies in Y. Let $x, y \in Y$ and let ω_{xy} be the shortest path in Y connecting them. Choose K to be the ball of radius $d(x,y)$ centered at x. If ω_{xy} is not a geodesic in X then there is z in the interior of ω_{xy} such that ω_{xy} is not a geodesic at z. We can shorten ω_{xy} by replacing a small subarc of ω_{xy} near z by the corresponding geodesic segment in X. By the remark at the beginning of the proof applied to $K = Y \cap B(x, d(x,y))$, the new curve is contained in Y. This is a contradiction. □

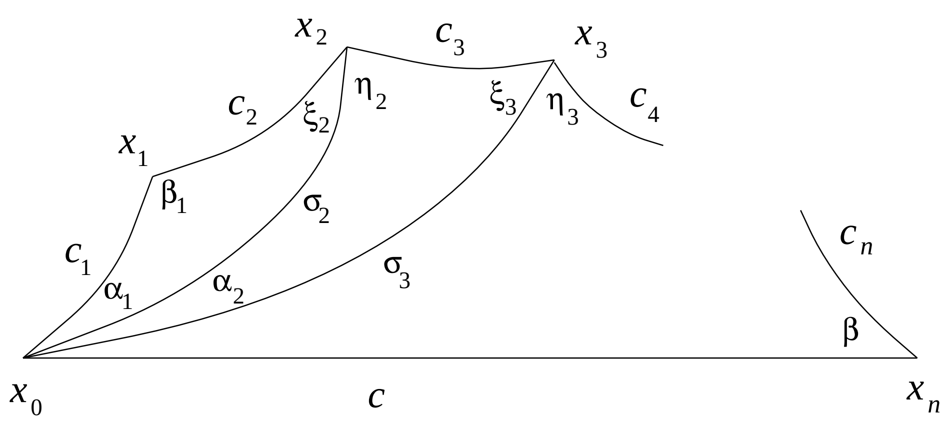

FIGURE 1

2.5 Lemma. *Let X be a simply connected space of nonpositive curvature, let $x_0, x_1, ..., x_n$ be pairwise distinct points in X and let c_i be the geodesic segments connecting x_{i-1} to x_i, $i = 1, 2, ..., n$. Let c connect x_0 to x_n and set $\alpha = \angle_{x_0}(c_1, c)$, $\beta = \angle_{x_n}(c, c_n)$ and $\beta_i = \angle_{x_i}(c_i, c_{i+1})$. Then*

$$\alpha + \beta \leq \sum_{i=1}^{n-1} (\pi - \beta_i).$$

Proof. Let σ_i be the geodesic from x_0 to x_i and let $\alpha_1 = \angle_{x_0}(c_1, \sigma_2)$, $\alpha_i = \angle_{x_0}(\sigma_i, \sigma_{i+1})$, $\xi_i = \angle_{x_i}(c_i, \sigma_i)$, $\eta_i = \angle_{x_i}(c_{i+1}, \sigma_i)$, $i = 2, ..., n-1$, see Figure 1. Then $\xi_i + \eta_i \geq \beta_i$ and $\alpha \leq \sum_{i=1}^{n-1} \alpha_i$. Since X has nonpositive curvature, the sum of the angles in each triangle $\Delta x_0 x_{i-1} x_i$ is at most π, and hence $\alpha_1 + \beta_1 + \xi_2 \leq \pi$, $\alpha_i + \eta_i + \xi_{i+1} \leq \pi$, $\alpha_{n-1} + \eta_{n-1} + \beta \leq \pi$. Adding these inequalities yields the lemma. □

Let $\phi : X \to X$ be an isometry of a Hadamard space. Then the **displacement function** $d_\phi(x) := d(x, \phi(x))$ is **convex**, that is, for any geodesic $\sigma : I \to X$ the function $d_\phi(\sigma(t))$ is convex in t. Isometries are classified according to the following possibilities for the displacement function. If d_ϕ achieves its minimum in X, then ϕ is called **semisimple**. If the minimum is 0, then ϕ has a fixed point and is called **elliptic**. If the minimum is positive and is achieved at $x \in X$, then the concatenation of the geodesic segments σ_i from $\phi^i(x)$ to $\phi^{i+1}(x)$, $i \in \mathbb{Z}$, is a geodesic σ which is invariant under ϕ and is called an **axis** of ϕ. In this case we call ϕ **axial**. If d_ϕ does not achieve a minimum, then ϕ is called **parabolic**. If Γ is a group of isometries of X acting properly discontinuously and cocompactly, then every $\phi \in \Gamma$ is semisimple, that is, either axial or elliptic.

We assume from now on that X is a locally compact Hadamard space. Our discussion of rank 1 spaces uses the following three lemmas from [Ba2] (see also [Ba1] for the case of Hadamard manifolds).

2.6 Lemma. *(See [Ba2], Lemma 3.1.1.) Let $\sigma : \mathbb{R} \to X$ be a unit speed geodesic which does not bound a flat strip of width $R > 0$.*

Then there are neighborhoods U of $\sigma(-\infty)$ and V of $\sigma(\infty)$ in $\overline{X}$ such that for any $\xi \in U$ and $\eta \in V$ there is a geodesic from ξ to η, and for any such geodesic σ' we have $d(\sigma', \sigma(0)) < R$. Moreover, σ' does not bound a flat strip of width $2R$. □

2.7 Lemma. *(See [Ba2], Lemma 3.1.2.) Let $\sigma : \mathbb{R} \to X$ be a unit speed geodesic which does not bound a flat half plane. Let (ϕ_n) be a sequence of isometries of X such that $\phi_n(x) \to \sigma(\infty)$ and $\phi_n^{-1}(x) \to \sigma(-\infty)$ for one (and hence any) $x \in X$.*

Then, for n sufficiently large, ϕ_n has an axis σ_n such that $\sigma_n(\infty) \to \sigma(\infty)$ and $\sigma_n(-\infty) \to \sigma(-\infty)$ as $n \to \infty$. □

2.8 Lemma. *(See [Ba2], Lemma 3.1.3.) Let ϕ be an isometry of X with an axis $\sigma : \mathbb{R} \to X$, where σ is a unit speed geodesic which does not bound a flat half plane. Then*

(i) *for any neighborhood U of $\sigma(-\infty)$ and neighborhood V of $\sigma(\infty)$ in $\overline{X}$ there exists $N \in \mathbb{N}$ such that $\phi^n(\overline{X} \setminus U) \subset V$, $\quad \phi^{-n}(\overline{X} \setminus V) \subset U \quad$ for all $n \geq N$;*

(ii) *for any $\xi \in X(\infty) \setminus \{\sigma(\infty)\}$ there is a geodesic σ_ξ from ξ to $\sigma(\infty)$, and any such geodesic does not bound a flat half plane. For any compact $K \subset X(\infty) \setminus \{\sigma(\infty)\}$, the set of these geodesics is compact (modulo parameterization).* □

We assume from now on that X is a locally finite simplicial complex with a piecewise smooth Riemannian metric g. We are interested in conditions under which X has nonpositive curvature in the sense defined above, that is $K_X \le 0$. We start by discussing the **links** of points $x \in X$. By subdividing X if necessary, we may assume that x is a vertex. Let A be a k-simplex adjacent to x. We view A as an affine simplex in $\mathbb{R}^k$, that is $A = \cap_{i=0}^k H_i$, where $H_0, H_1, ..., H_k$ are closed half spaces in general position and WLOG $x \in \operatorname{Int}(H_0)$. The Riemannian metric g_A is the restriction to A of a smooth Riemannian metric defined in an open neighborhood V of A in $\mathbb{R}^k$. The intersection

$$T_x A := \cap_{i=1}^k H_i \subset T_x V$$

is a cone with apex $0 \in T_x V$, and $g_A(x)$ turns it into a Euclidean cone. Let $B \subset A$ be another simplex adjacent to x. Then the face of $T_x A$ corresponding to B is isomorphic to $T_x B$ and we view $T_x B$ as a subset of $T_x A$. Set

$$T_x X = \cup_{A \ni x} T_x A\,.$$

Let $S_x A$ denote the subset of all unit vectors in $T_x A$ and set

$$S_x = S_x X = \cup_{A \ni x} S_x A\,.$$

The set S_x is called the **link** of x in X (or the **space of directions**). If A is a k-simplex adjacent to x, then $g_A(x)$ defines a Riemannian metric on the $(k-1)$-simplex $S_x A$. The family g_x of Riemannian metrics $g_A(x)$ turns $S_x X$ into a simplicial complex with a piecewise smooth Riemannian metric such that the simplices are spherical: a k-simplex in S_x is (isometric to) the intersection of $k+1$ closed hemispheres in S^k in general position. We denote by d_x the associated metric.

Suppose now that the dimension of X is 2. If x lies in the interior $\mathring{f}$ of a face f, then $S_x X$ is the unit circle of the smooth surface $\mathring{f}$ with respect to the Riemannian metric $g_f | \mathring{f}$. If x lies in the interior $\mathring{e}$ of an edge e, then $S_x X$ is the bipartite graph with two vertices corresponding to the directions of e at x and edges of length $\pi/2$ which represent the faces adjacent to e and connect the two vertices. If x is a vertex, then $S_x X$ is a graph whose vertices correspond to the edges adjacent to x. Two such vertices ξ and η of $S_x X$ are connected by an edge of length $\alpha_f \in (0, \pi)$ if the corresponding edges e_ξ and e_η of X are adjacent to a face f with interior angle α_f at x (see Figure 2).

2.10 Theorem. *(See [BaBu]). Let g be a piecewise smooth Riemannian metric on a locally finite two-dimensional simplicial complex X and let d be the associated distance function.*

Then $K_X \le K$ iff the following three conditions hold:

(i) *the Gauss curvature of the open faces is bounded from above by K;*
(ii) *for every edge e of X, every two faces f_1, f_2 adjacent to e and every interior point $x \in e$ the sum of the geodesic curvatures $k_1(x), k_2(x)$ of e with respect to f_1, f_2 is nonpositive;*
(iii) *for every vertex v of X every simple loop in $S_x X$ has length at least 2π.* $\square$

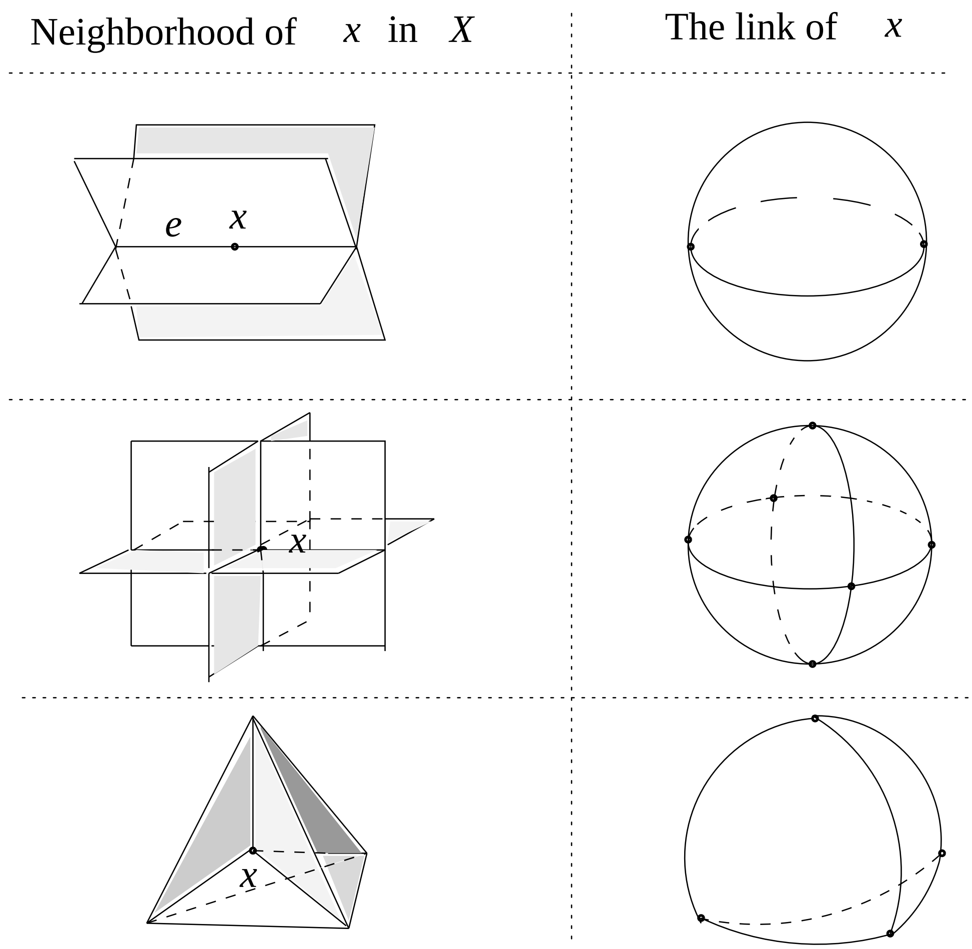

FIGURE 2

Let X be a 2-dimensional simplicial complex without boundary. An edge of X is called **inessential** if it bounds exactly two faces; the other edges are called **essential**. A vertex v of X is called **inessential** if its link is homeomorphic to a bipartite graph with two vertices and $m(v) \geq 0$ edges connecting them; the other vertices are called **essential**. An inessential vertex is called **interior** if its link is homeomorphic to the circle (that is, if $m(v) = 2$).

A connected component of the union of all open faces, inessential edges and interior vertices is called a **maximal face** of X. A connected component of the union of all open essential edges and inessential but not interior vertices v is called a **maximal essential edge**. A maximal essential edge might be a loop. Since X is boundaryless, the maximal faces of X are bounded by maximal essential edges and essential vertices.

2.11 Proposition. *Let (X, Γ) be a compact 2-dimensional boundaryless orbihedron with a metric d of nonpositive curvature. Assume that d is induced by a Γ-invariant piecewise smooth Riemannian metric with respect to a Γ-invariant triangulation of X such that*

(i) *the Gauss curvature of all faces is* 0*;*

(ii) *for every edge e and any two faces f_1, f_2 adjacent along e and for any $x \in e$ we have $k_1(x) + k_2(x) = 0$;*

(iii) *for every interior vertex v, the complete angle of S_v is 2π.*

Then the maximal essential edges of X are geodesics and the maximal faces of X are smooth flat surfaces. Moreover, X admits a Γ-invariant triangulation such that all edges are geodesics and all faces are Euclidean triangles.

Proof. If e is an essential edge and if e is not adjacent to any face, then e is a geodesic. If e is adjacent to at least three faces, then the geodesic curvature at e with respect to any face adjacent to e is 0 since, by (ii), the sum of the geodesic curvatures of every pair of these faces is 0. Hence all essential edges are geodesics.

Assume now that e is an inessential edge and that f_1, f_2 are the two faces adjacent to e. Since the metrics on f_1 and f_2 are flat and since the geodesic curvatures k_1 and k_2 of e with respect to f_1 and f_2 add up to 0 precisely, the metrics extend smoothly to a flat metric on $f_1 \cup e \cup f_2$. (View e, f_1 and f_2 locally as sitting in the Euclidean plane.) Now (iii) implies that the maximal faces of X are smooth flat surfaces.

We now first replace the inessential edges of X in a Γ-equivariant way by piecewise geodesics arcs. Then we subdivide the faces Γ-equivariantly so that the break points of these arcs become vertices. □

2.12 Proposition. *Let (X, Γ) be a compact 2-dimensional orbihedron with a piecewise smooth metric of nonpositive curvature.*

Then (X, Γ) contains a Γ-invariant suborbihedron X' without boundary such that

(i) *X' is a Γ-equivariant strong deformation retract of X;*

(ii) *the action of Γ on X' is effective;*

(iii) *the induced metric on X' is piecewise smooth and of nonpositive curvature.*

Proof. Fix a Γ-invariant triangulation on X such that the given metric is piecewise smooth with respect to it. Iteratively, we apply the following reductions:

(a) Delete boundary vertices and the unique open edges adjacent to them.

(b) If f is an open face adjacent to exactly one open boundary edge e, then delete f and e.

(c) If f is an open face adjacent to exactly two open boundary edges e_1 and e_2, then replace $f \cup e_1 \cup e_2$ by the segment from the midpoint of the third edge to the opposite vertex.

(d) If f is an open face with three open boundary edges e_1, e_2, e_3, then replace $f \cup e_1 \cup e_2 \cup e_3$ by the three segments from the barycenter of f to the vertices.

We may have to apply each of these reductions more than once. Since X has only finitely many simplices mod Γ, and since these reductions do not increase their number, after a finite number of steps we end up with a boundaryless complex X', as asserted. □

3. Liouville measure for the geodesic flow

From now on we assume that X is a locally finite, n-dimensional and boundaryless complex with a piecewise smooth Riemannian metric. We denote by $X^{(k)}$ the k-skeleton of X and by X' the set of $x \in X$ such that x is contained in the interior of an $(n-1)$-simplex adjacent to at least two n-simplices.

Let $x \in X'$. Then x is contained in the interior of an $(n-1)$-simplex B. For any n-simplex C whose boundary ∂C contains x let $S'_x C$ denote the open hemisphere of unit tangent vectors at x pointing inside C. Let $C_1, ..., C_m$, $m \geq 2$, be the n-simplices containing B. We set

$$S'_x = \bigcup_{i=1}^{m} S'_x C_i, \quad S' = \bigcup_{x \in X'} S'_x \quad \text{and} \quad S'C = \bigcup_{x \in \partial C \cap X'} S'_x C.$$

For $v \in S'_x C$ denote by $\theta(v)$ the angle between v and the interior normal $\nu_C(x)$ of B with respect to C at x. Let dx be the volume element on X' and let λ_x be the Lebesgue measure on S'_x. We define the Liouville measure on S' by

$$d\mu(v) = \cos\theta(v)\, d\lambda_x(v)\, dx. \tag{3.1}$$

Note that $d\mu(v) \times dt$ is the ordinary Liouville measure invariant under the geodesic flow on each n-simplex C of X. Therefore, for μ-a.e. $v \in S'C$ the geodesic γ_v in C determined by $\dot\gamma_v(0) = v$ meets $\partial C \cap (X^{(n-1)} \setminus X^{(n-2)})$ after a finite time $t_v > 0$ so that $I(v) \stackrel{\text{def}}{=} -\dot\gamma_v(t_v) \in S'C$, see Figure 3. Note that $\gamma_v(t_v) \in X'$ since X is boundaryless. Similarly to the billiard flow, μ is invariant under the involution I (see, for example, [CFS]).

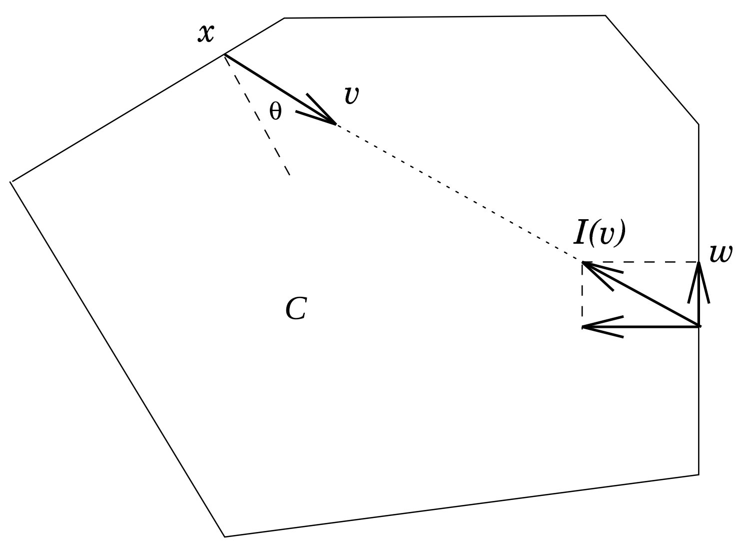

Figure 3

Let $I(v) = w + \cos\theta(I(v))\,\nu_C(\gamma_v(t_v))$, where w is tangent to X' and set

$$F(v) = \bigcup_{C'}(-w + \cos\theta(I(v))\,\nu_{C'}(\gamma_v(t_v))),$$

where the union is taken over all n-simplices C' containing $\gamma_v(t_v)$ except C.

Thus there is a subset $S_1 \subset S'$ of full μ-measure such that $F(v)$ is defined for any $v \in S_1$. We set recursively $S_{i+1} = \{v \in S_1 : F(v) \subset S_i\}$ and define $S_\infty = \cap_{i=1}^{\infty} S_i$, $V = S_\infty \cap I(S_\infty)$. By construction, V has full μ-measure. We define now the transition probabilities for a Markov chain with states in V by the formula

$$p(v,w) = \begin{cases} \dfrac{1}{|F(v)|} & \text{if } w \in F(v) \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

where $|F(v)|$ is the cardinality of $F(v)$.

3.3 Proposition. *Let X be a locally finite boundaryless simplicial complex with a piecewise smooth metric g.*

Then the measure μ given by (3.1) is stationary for the Markov chain on V with transition probabilities $p(v,w)$ given by (3.2).

Proof. For $w \in F(v)$ set $q(I(v),w) = (p(v,w))^{-1}$ and $H(I(v)) = F(v)$. Let $P(v,M)$ denote the transition probability from $v \in V$ to a measurable subset $M \subset V$. Then we have

$$\begin{aligned} \int_V P(v,M)\,d\mu(v) &= \int_V \sum_{w\in F(v)} p(v,w)\,\chi_M(w)\,d\mu(v) \\ &= \int_V \sum_{w\in F(v)} q^{-1}(I(v),w)\,\chi_M(w)\,d\mu(v). \end{aligned}$$

Since I preserves μ, the last expression is equal to

$$\int_V \sum_{w\in H(u)} q^{-1}(u,w)\,\chi_M(w)\,d\mu(u) = \sum_B \int_{V(B)} \sum_{w\in H(u)} q^{-1}(u,w)\,\chi_M(w)\,d\mu(u),$$

where $V(B)$ denotes the set of vectors from V with foot point in B. Note that for $w \in H(u)$ the number $q(u,w)$ is exactly the cardinality of $H(u)$ or the number of terms in the inner sum. Hence the last expression is equal to

$$\sum_B \int_{V(B)} \chi_M(w)\,d\mu(w).$$

□

Let V^* be the set of sequences $(v_n)_{n\in\mathbb{Z}} \subset V$ such that $v_{n+1} \in F(v_n)$ for all $n \in \mathbb{Z}$. Proposition 3.3 implies that μ induces a shift invariant measure μ^* on V^*.

Recall that $G(X)$ denotes the space of complete geodesics in X and the geodesic flow $\{g^t\}$ in $G(X)$ acts by the formula

$$(g^t\sigma)(s) = \sigma(s+t)\,.$$

Let $G^*(X) \subset G(X)$ denote the set of geodesics which do not meet $X^{(n-2)}$ and intersect $(n-1)$-simplices transversally. Then $G^*(X)$ is a Borel subset of full μ-measure in $G(X)$ and we may think of V^* as a cross section for the geodesic flow on $G^*(X)$. Thus the measure μ^* on V^* defines a measure on $G^*(X)$ invariant under the geodesic flow. We call this measure the **Liouville measure** since it is the natural generalization of the usual Liouville measure on the unit tangent bundle of a smooth Riemannian manifold.

3.4 Remarks. a) The set $G^*(X)$ is not dense in $G(X)$ if there is a vertex whose diameter is $> \pi$.
b) The Liouville measure μ is positive on nonempty open subsets of $G^*(X)$.

If Γ is a group of isometric automorphisms of X that acts cocompactly and properly discontinuously, then the Liouville measure defines a finite invariant measure for the geodesic flow in $G^*(X)/\Gamma$. Recall that a geodesic $\sigma \in G(X)$ is Γ-**recurrent** if there are isometries $\phi_n \in \Gamma$ and $t_n \to \infty$ for which $\phi_n(g^{t_n}(\sigma)) \to \sigma$ as $n \to \infty$.

3.5 Corollary. *Let (X,Γ) be a compact orbihedron without boundary and with a piecewise smooth metric. Then*

(i) For every subset $G \subset G^*(X)$ with $\mu(G) > 0$ and any $T > 0$ there are $\sigma \in G$, $\phi \in \Gamma$ and $t \geq T$ such that $\phi(g^t\sigma) \in G$.

(ii) with respect to the Liouville measure, almost every geodesic in $G(X)$ is Γ-recurrent.

Proof. The statement follows directly from the Poincaré recurrence theorem for the induced action of g^t on $G(X)/\Gamma$. □

4. Some Properties of Rank 1 Spaces

Let (X, Γ) be a compact orbispace of nonpositive curvature. Then X is a locally compact Hadamard space, that is a locally compact, simply connected, complete geodesic space of nonpositive curvature. Observe that if $\sigma : \mathbb{R} \to X$ is a geodesic that does not bound a flat half plane then there is $R > 0$ such that σ does not bound a flat strip of width R.

We say that $\xi, \eta \in X(\infty)$ are **dual** (relative to Γ) if for any neighborhoods U of ξ and V of η in $\overline{X}$ there is $\psi \in \Gamma$ such that

$$\psi(\overline{X} \setminus U) \subset V \text{ and } \psi^{-1}(\overline{X} \setminus V) \subset U\,.$$

The set Δ_ξ of points $\eta \in X(\infty)$ dual to $\xi \in X(\infty)$ is clearly closed and Γ-invariant. Lemma 2.8 implies that the endpoints of a Γ-closed geodesic of rank 1 are dual.

4.1 Theorem. *Let (X, Γ) be a compact orbispace of rank 1 and assume that $X(\infty)$ contains more than two points.*

Then $X(\infty)$ is a perfect set and any point in $X(\infty)$ is dual to any other point and to itself. Moreover, for any two nonempty open subsets $U, V \subset X(\infty)$ there is a Γ-closed geodesic ω of rank 1 with $\omega(-\infty) \in U$ and $\omega(\infty) \in V$.

4.2 Remarks. a) Note that $X(\infty)$ consists of at least two points, and if $|X(\infty| = 2$, then X is quasi-isometric to the real line and Γ contains an infinite cyclic subgroup of finite index (given by the powers of the rank 1 isometry).

b) Following Chen and Eberlein [ChEb] we say that Γ satisfies the **duality condition** if for any geodesic $\sigma : \mathbb{R} \to X$ there is a sequence $\phi_n \in \Gamma$ such that $\phi_n(x) \to \sigma(\infty)$ and $\phi_n^{-1}(x) \to \sigma(-\infty)$ as $n \to \infty$ for any point $x \in X$. Theorem 4.1 and the remark above imply that Γ satisfies the duality condition if (X, Γ) is a compact orbihedron of rank 1.

In order to prove Theorem 4.1 we begin with three lemmas. Fix an isometry $\phi \in \Gamma$ translating a geodesic σ that does not bound a flat half plane. Set $\sigma(0) = x_0$ and $\sigma(t_0) = \phi(x_0)$, where t_0 is the period of σ, and let $R_0 > 0$ be such that σ does not bound a flat strip of width R_0. Set $\overline{\sigma} = \sigma \cup \{\sigma(\infty), \sigma(-\infty)\}$.

4.3 Lemma. *For any $T, \varepsilon > 0$ there is $R > 0$ such that for any $x \in X$ with $d(x, \sigma) > R$ and any two points $y, z \in \overline{\sigma}$ the unit speed geodesics γ_{xy} and γ_{xz} connecting x with y and z satisfy*

$$d(\gamma_{xy}(t), \gamma_{xz}(t)) \le \varepsilon,\ 0 \le t \le T\,.$$

Proof. Since σ is invariant under ϕ it suffices to consider only those $x \in X$ for which the closest point of σ lies in $\sigma([0, t_0])$. These points x form a compact subset A of $\overline{X}$ and $\sigma(\pm\infty) \notin A$, see Figure 4.

Choose neighborhoods U of $\sigma(-\infty)$ and V of $\sigma(\infty)$ in $\overline{X}$ so that any geodesic from U to V passes through the ball $B(x_0, R_0)$, see Lemma 2.6. By Lemma 2.8 there is $N \in \mathbb{N}$ such that $\phi^{-n}(A) \subset U$ and $\phi^n(A) \subset V$ for all $n \ge N$. Choose $t_1 > 0$ such that $\sigma([-\infty, -t_1]) \subset U$ and $\sigma([t_1, \infty]) \subset V$. Hence for any point $x \in \phi^{-n}(A)$ the geodesic γ_{xy} connecting x with

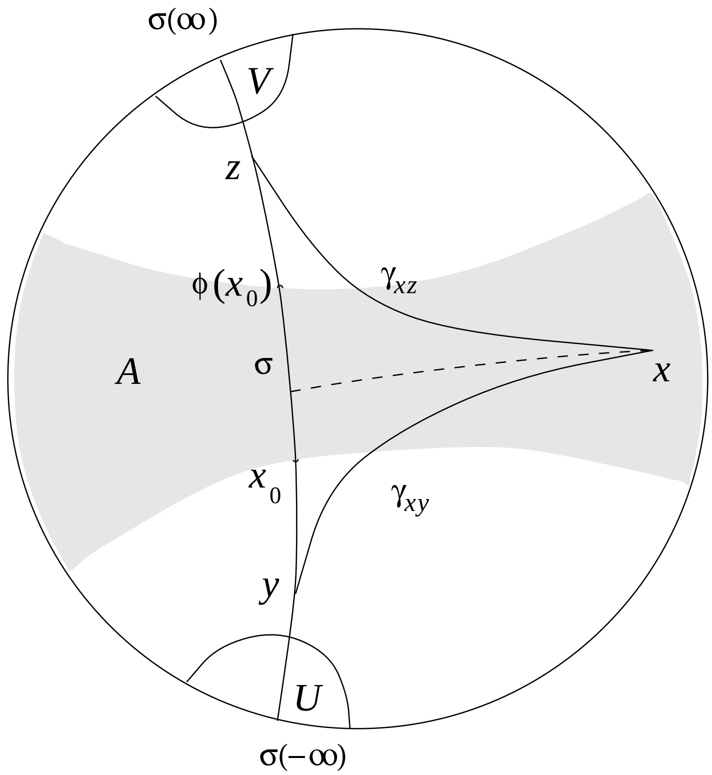

FIGURE 4

$y \in \sigma([t_1, \infty])$ passes through $B(x_0, R_0)$. Since $d(x_0, \phi^{-n}(A)) \to \infty$, for a sufficiently large n any two such geodesics γ_{xy} and γ_{xz}, by comparison with the plane, satisfy

$$d(\gamma_{xy}(t), \gamma_{xz}(t)) \leq \varepsilon/3, \;\; 0 \leq t \leq T.$$

Hence the statement of the lemma holds for any $x \in A$ and $y, z \in \sigma([t_1 + nt_0, \infty])$ with $\varepsilon/3$ instead of ε. Similarly, for a large enough n the same estimate holds for $x \in A$ and $y, z \in \sigma([-\infty, -t_1 - nt_0])$. Using comparison with the plane we obtain the necessary estimate for the middle segment $\sigma([-t_1 - nt_0, t_1 + nt_0])$ for a large enough R. □

4.4 Lemma. *For any $T, \varepsilon > 0$ there is $R' > 0$ such that if $d(x, x_0) > R'$ then*

$$d(\gamma_{xx_0}(t), \gamma_{x\sigma(\infty)}(t)) \leq \varepsilon, \quad 0 \leq t \leq T,$$
$$\textit{or} \quad d(\gamma_{xx_0}(t), \gamma_{x\sigma(-\infty)}(t)) \leq \varepsilon, \quad 0 \leq t \leq T.$$

Proof. Choose neighborhoods U of $\sigma(-\infty)$ and V of $\sigma(\infty)$ in $\overline{X}$ so that any geodesic from U to V passes through the ball $B(x_0, R_0)$. By decreasing U and V if necessary we may assume that the first inequality holds for all $x \in U$ and the second one holds for all $x \in V$. Now let R be from Lemma 4.3 and choose R' so large that $d(x, \sigma) > R$ if $x \notin U \cup V \cup B(x_0, R')$. □

4.5 Lemma. *Let ψ_i be an axial isometry with an axis σ_i, $i = 1, 2$. Assume that $\sigma_1(-\infty) = \sigma_2(-\infty)$. Then $\sigma_1(\infty) = \sigma_2(\infty)$.*

Proof. Let $x_i = \sigma_i(0)$, $i = 1, 2$. For every $n > 0$ there is m such that $d(\psi_2^m \psi_1^{-n} x_1, x_2)$ does not exceed the sum of the period of σ_2 and $d(x_1, x_2)$. Since Γ acts properly discontinuously, there is $\psi_0 \in \Gamma$ such that $\psi_2^m \psi_1^{-n} = \psi_0$ for infinitely many pairs m, n. Therefore, $\psi_1^m = \psi_2^n$ for some $m, n \neq 0$. □

Proof of Theorem 4.1. Since $X(\infty)$ contains more than two points and since Γ acts cocompactly, there is $\psi_0 \in \Gamma$ such that $d(\psi_0(x_0), \sigma) > R_0$. The geodesic $\psi_0(\sigma)$ does not bound a flat half plane and is an axis of $\psi_0 \phi \psi_0^{-1} \in \Gamma$. Therefore the points $\psi_0(\sigma(\pm\infty))$ are dual to each other. Since σ does not belong to a flat strip of width greater than R_0, we have, by Lemma 4.5, that $\psi_0(\sigma(\pm\infty)) \notin \{\sigma(\infty), \sigma(-\infty)\}$. By Lemma 2.8, $\phi^n(\psi_0(\sigma(\pm\infty))) \to \sigma(\infty)$ as $n \to \infty$, and hence $\sigma(\infty)$ is dual to both $\psi_0(\sigma(\infty))$ and $\psi_0(\sigma(-\infty))$. By symmetry, each of the four points $\sigma(\pm\infty)$, $\psi_0(\sigma(\pm\infty))$ is dual to every other and to itself.

Now let $\xi \in X(\infty)$, $\xi \neq \sigma(\pm\infty)$ and let $\psi_n \in \Gamma$ be such that $\psi_n x \to \xi$ for any $x \in X$. By Lemma 4.4, $\psi_n(\sigma(\infty)) \to \xi$ or $\psi_n(\sigma(-\infty)) \to \xi$ (or both) and hence ξ is dual to $\sigma(-\infty)$ and $\sigma(\infty)$. Now let η be any other point in $X(\infty)$. Choose $\phi_n \in \Gamma$ so that $\phi_n(\sigma(\infty)) \to \eta$ or $\phi_n(\sigma(-\infty)) \to \eta$. Since ξ is dual to both $\sigma(\infty)$ and $\sigma(-\infty)$ we conclude that ξ is dual to η. Hence any two points in $X(\infty)$ are dual. Clearly $X(\infty)$ is a perfect set.

To prove the last assertion of the theorem, let $U, V \subset X(\infty)$ be nonempty open subsets. By Lemma 2.8, there is a geodesic ω_1 from $\sigma(-\infty)$ to a point $\xi \in U$ such that ω_1 does not bound a flat half plane. Since $\sigma(-\infty)$ is dual to ξ, there is a sequence $\phi_n \in \Gamma$ such that $\phi_n(x) \to \xi$ and $\phi_n^{-1}(x) \to \sigma(-\infty)$ for any $x \in X$ as $n \to \infty$. By Lemma 2.7, if n is sufficiently large, then ϕ_n has an axis σ_n such that $\eta = \sigma_n(\infty) \in U$ and σ_n does not bound a flat half plane. By Lemma 2.8, there is a geodesic ω_2 from η to a point $\zeta \in V$ such that ω_2 does not bound a flat half plane. Now η and ζ are dual and Lemma 2.7 implies the existence of a Γ-closed geodesic of rank 1 with endpoints in U and V. □

We derive some applications of Theorem 4.1 which are relevant in our paper. Applications to random walks on Γ can be found in [Ba2].

4.6 Theorem. *Let (X, Γ) be a compact orbispace of rank 1 and assume that $X(\infty)$ contains more than two points.*

Then Γ contains a free nonabelian subgroup.

Proof. Choose disjoint open subsets $U, V \in X(\infty)$ with $U \cup V \neq X(\infty)$ and let $\xi \in X(\infty) \setminus (U \cup V)$. By Theorem 4.1 there are $\phi, \psi \in \Gamma$ such that

$$\phi^{\pm 1}(X(\infty) \setminus U) \subset U \quad \text{and} \quad \phi^{\pm 1}(X(\infty) \setminus V) \subset V.$$

Let w be a nontrivial reduced word in ϕ and ψ. Since $U \cap V = \emptyset$ we conclude that $w(\xi) \in U$ if w starts with a power of ϕ and that $w(\xi) \in V$ if w starts with a power of ψ. In either case $w(\xi) \neq \xi$, hence $w \neq id$. Therefore ϕ and ψ generate a free subgroup of Γ. □

We say that X is **geodesically complete** if any geodesic segment in X is contained in a complete geodesic.

4.7 Theorem. *Let (X, Γ) be a compact, geodesically complete orbispace of rank 1 and assume that $X(\infty)$ contains more that two points.*

Then the geodesic flow of X is topologically transitive mod Γ, that is, for any two nonempty open subsets $U, V \subset G(X)$ there are $t \in \mathbb{R}$ and $\phi \in \Gamma$ with $g^t(U) \cap \phi(V) \neq \emptyset$.

4.8 Remarks. (a) Topological transitivity is equivalent to the existence of an orbit of (g^t) which is dense mod Γ.

(b) In general, the geodesic flow is not topologically mixing mod Γ.

Proof of Theorem 4.7. We let $U(\infty)$ (respectively $V(\infty)$) be the set of points $\sigma(\infty)$ in $X(\infty)$ with $\sigma \in U$ (respectively $\sigma \in V$). Then $U(\infty)$ and $V(\infty)$ are open and nonempty. By Theorem 4.1, we can assume that $U(\infty) \subset V(\infty)$. Let $\sigma \in U$. Then there is a geodesic $\sigma' \in V$ with $\sigma'(\infty) = \sigma(\infty)$. Since U and V are open there are $\varepsilon > 0$ and $T > 0$ such that a geodesic $\tilde{\sigma}$ belongs to U if

$$d(\sigma(T), \tilde{\sigma}(T)),\ d(\sigma(-T), \tilde{\sigma}(-T)) < \varepsilon$$

and belongs to V if

$$d(\sigma'(T), \tilde{\sigma}(T)),\ d(\sigma'(-T), \tilde{\sigma}(-T)) < \varepsilon.$$

Now let $x = \sigma(T)$ and $x' = \sigma'(-T)$ and choose a sequence (ϕ_n) in Γ such that $\phi_n(y) \to \sigma(\infty)$ and $\phi_n^{-1}(y) \to \sigma(-\infty)$ for any $y \in X$. Let σ_n be a unit speed geodesic with $\sigma_n(-T) = x'$ and $\sigma_n(t_n) = \phi_n(x)$, where $t_n = d(x', \phi_n(x)) - T$. Then σ_n is in V for n sufficiently large since $\phi_n(x) \to \sigma'(\infty) = \sigma(\infty)$. Furthermore $\phi_n^{-1}(g^{t_n}\sigma_n') \in U$ for n sufficiently large since $\phi_n^{-1}(g^{t_n}\sigma_n')(T) = x$ and $\phi_n^{-1}(x') \to \sigma(-\infty)$. □

Recall that a compact 2-dimensional boundaryless orbihedron (X, Γ) with a piecewise smooth metric of nonpositive curvature is geodesically complete. Thus assertion (ii) of Theorem D in the Introduction is a special case of Theorem 4.7. We now prove assertion (i) of Theorem D.

4.9 Theorem. *Let (X, Γ) be a closed 2-dimensional rank 1 boundaryless orbihedron with a piecewise smooth metric of nonpositive curvature.*

Then hyperbolic Γ-closed geodesics are dense in the space of geodesics.

Proof. Consider the set U of geodesics σ such that a) σ passes through a point in an open face where the Gauss curvature is negative, or b) σ passes from one face to another through a point in an open edge e where the sum of the geodesic curvatures of e with respect to the two faces is negative, or c) σ passes through a vertex v and the distance between the incoming and outgoing direction of σ in the link of v is $> \pi$. Then U is open and invariant under the geodesic flow. Now U is not empty since (X, Γ) has rank 1. Since the geodesic flow is topologically transitive, U is dense in the space of geodesics. By Lemmas 2.6 and 2.7, each geodesic in U is a limit of hyperbolic Γ-closed geodesics. □

5. Homotopy invariance of rank 1

5.1 Definition. Let X be a metric space and let $A \geq 1$, $B \geq 0$. A curve $c : I \to X$ is an (A, B)-**quasigeodesic** in X if for all $s, t \in I$

$$A^{-1}|s-t| - B \leq d(c(s), c(t)) \leq A|s-t| + B\,.$$

Note that we do not assume that a quasigeodesic is a continuous curve.

5.2 Theorem. *Let X be a locally compact Hadamard space having an axial isometry ϕ with an axis σ. Then σ does not bound a flat half plane if and only if for any $A \geq 1$, $B \geq 0$ there is $R = R(\sigma)$ such that any (A, B)-quasigeodesic c with ends on σ stays in the R-neighborhood of σ.*

If σ bounds a flat half plane then clearly such an R does not exist. In what follows we assume that σ does not bound a flat half plane. We will need the following three lemmas. Let P denote the projection to σ.

5.3 Lemma. *There are constants $R_0, T > 0$ such that if $x, y \in X$, $d(x, \sigma), d(y, \sigma) \geq R_0$ and $d(Px, Py) \geq T$ then*

$$d(x, y) \geq d(Px, Py) + 1\,.$$

Proof. If this is not so, there are points $x_n, y_n \in X$ with $d(x_n, \sigma)$,$d(y_n, \sigma)$, $d(Px_n, Py_n) \geq n$ and $d(x_n, y_n) < d(Px_n, Py_n)+1$. Fix a positive integer m. By choosing n large enough and applying an appropriate power of ϕ, we may assume that the segment of σ between Px_n and Py_n contains $\sigma([-m, m])$. WLOG assume that the points $Px_n, \sigma(-m), \sigma(m), Py_n$ lie in this order on σ. Let σ_n be the geodesic connecting x_n to y_n and denote by $p_n, q_n \in \sigma_n$ the points for which $Pp_n = \sigma(-m)$, $Pq_n = \sigma(m)$. By Corollary 2.3, P does not increase distances. Hence,

$$d(x_n, p_n) \geq d(Px_n, \sigma(-m)) \quad \text{and} \quad d(y_n, q_n) \geq d(Py_n, \sigma(-m))\,.$$

We have

$$d(x_n, y_n) = d(x_n, p_n) + d(p_n, q_n) + d(q_n, y_n)\,.$$

Therefore,

$$d(p_n, q_n) \leq d(x_n, y_n) - d(Px_n, \sigma(-m)) - d(Py_n, \sigma(m))\,.$$

Hence

$$d(p_n, q_n) \leq d(\sigma(-m), \sigma(m)) + 1 = 2m + 1\,. \tag{5.4}$$

Let α_n, β_n be the geodesics from $\sigma(-m)$ to p_n and from $\sigma(m)$ to q_n respectively. By passing to a subsequence if necessary, we may assume that α_n and β_n converge to geodesic rays α and β. By construction, the angles between α and σ, β and σ are both at least $\pi/2$. Hence $d(\alpha(t), \beta(t))$ is not decreasing. By (5.4), $d(\alpha(t), \beta(t)) \leq 2m + 1$ for all $t \geq 0$. Hence α, β and σ bound a flat half strip S_m with right angles at $\sigma(\pm m)$. For a subsequence $m_k \to \infty$ the corresponding half strips S_{m_k} converge to a flat half plane along σ. Contradiction. □

5.5 Lemma. *Let T be from Lemma 5.3. Then for any $K \geq 1$ there is $R_1 > 0$ such that $d(x,y) \geq KT$ provided $d(x,\sigma), d(y,\sigma) \geq R_1$ and $d(Px,Py) \geq T$.*

Proof. By Lemma 5.3 and the convexity of the distance, we have

$$d(x,y) \geq d(Px,Py) + m \geq T + m \quad \text{if} \quad d(x,\sigma), d(y,\sigma) \geq mR_0,$$

where $m \geq 1$. Now choose $m = [(K-1)T] + 1$ and $R_1 = mR_0$. $\square$

5.6 Lemma. *Let $c : [u,v] \to X$ be an (A,B)-quasigeodesic. Assume that for every $t \in [u,v]$ we have $d(c(t),\sigma) \geq R_1$, where R_1 is chosen by Lemma 5.5 for $K > 2B/T$. Let $Pc(u) = \sigma(t_u)$ and $Pc(v) = \sigma(t_v)$ and assume that $\sigma([t_u,t_v])$ contains a segment $\sigma(t_0, t_0 + T)$ of length T. Then there is $s_0 \in (u,v)$ with $Pc(s_0) \in \sigma((t_0,t_0+T))$.*

Proof. If there is no such s_0 then $(u,v) = U \cup V$, where

$$U = \{s : Pc(s) = \sigma(t),\, t \leq t_0\},\ V = \{s : Pc(s) = \sigma(t),\, t \geq t_0 + T\}.$$

Note that by Lemma 5.5 and the choice of K and R_1, $|s_u - s_v| \geq \dfrac{KT - B}{A} \geq \dfrac{B}{A}$, whenever $s_u \in U$, $s_v \in V$. Contradiction. $\square$

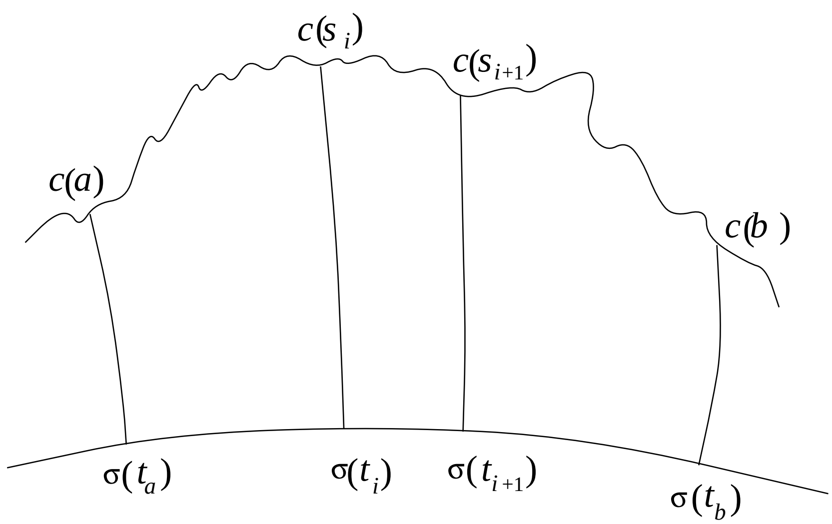

FIGURE 5

Proof of Theorem 5.2. Set $K = \max(3A^2B/T,\, 25A^4)$ and choose R_1 by Lemma 5.5. Assume that there is $\tilde{s}$ such that $d(c(\tilde{s}),\sigma) \geq R_1 + B$. Set

$$a = \sup\{s \leq \tilde{s} : d(c(s),\sigma) < R_1 + B\},\ b = \inf\{s \geq \tilde{s} : d(c(s),\sigma) < R_1 + B\}.$$

Then $d(c(s),\sigma) \geq R_1 + B$ for $s \in (a,b)$, and hence

$$R_1 \leq d(c(a),\sigma), d(c(b),\sigma) \leq R_1 + 2B$$

since c is an (A,B)-quasigeodesic. Let $Pc(a) = \sigma(t_a)$, $Pc(b) = \sigma(t_b)$ and assume that $t_a \leq t_b$. Let $m \geq 0$ be such that $t_b - t_a = 2mT + \tau$, where $0 \leq \tau < 2T$. By Lemma 5.6, there are $a = s_0 < s_1 < s_2 < \ldots < s_{m+1} = b$ such that $Pc(s_i) = \sigma(t_i)$ with $t_i \in (t_a + (2i-1)T, t_a + 2iT)$, $1 \leq i \leq m$, see Figure 5. By Lemma 5.5 and since c is a quasigeodesic, we have

$$A(s_i - s_{i-1}) + B \geq d(c(s_i), c(s_{i-1})) \geq KT\,,$$

and hence, $B \leq KT - B \leq A(s_i - s_{i-1})$, $1 \leq i \leq m$. Therefore,

$$\begin{aligned}(5.7)\qquad & \sum_{i=1}^{m+1} d(c(s_i), c(s_{i-1})) \leq \sum_{i=1}^{m+1} (A(s_i - s_{i-1}) + B) \\ & \leq A(b-a) + A(s_m - a) + B \leq 2A(b-a) + B\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}& \sum_{i=1}^{m} d(c(s_i), c(s_{i-1})) \geq \sum_{i=1}^{m} (A^{-1}(s_i - s_{i-1}) - B) \\ & \geq m(\frac{KT - B}{A^2} - B) \geq \frac{1}{3} m \frac{KT}{A^2} \geq \frac{1}{3}\frac{K}{A^2}\frac{1}{2}(t_b - t_a - 2T)\,,\end{aligned}$$

which by the triangle inequality is

$$\geq \frac{1}{6}\frac{K}{A^2}(d(c(a),c(b)) - 2R_1 - 4B - 2T) \geq \frac{1}{6}\frac{K}{A^3}((b-a) - AB - 2AR_1 - 4AB - 2AT)$$

and if $b - a > 2(2AR_1 + 5AB + 2AT) \stackrel{\text{def}}{=} \kappa$ the latter is

$$\geq \frac{1}{12A^3} K(b-a) > 2A(b-a) + B\,,$$

which contradicts (5.7). Hence $b - a \leq \kappa$. Since c is a quasigeodesic, $d(c(s), c(a)) \leq A\kappa + B$ for all $s \in [a,b]$ and hence

$$d(c(s),\sigma) \leq R_1 + 3B + A\kappa \stackrel{\text{def}}{=} R\,.$$

□

Let Γ be a finitely generated discrete group and let d_Γ be the word metric on Γ corresponding to a finite system of generators. Note that any two word metrics on Γ are equivalent and the notion of quasigeodesics (see Definition 5.1) can be applied to Γ.

5.8 Definition. A finitely generated discrete group Γ has **rank 1** if there is $\phi \in \Gamma$ with the property that $\Phi = \{\phi^k | k \in \mathbb{Z}\}$ is a quasigeodesic and for any $A \geq 1$, $B \geq 0$ there exists $R > 0$ such that any (A, B)-quasigeodesic $c : [a, b] \to \Gamma$ with endpoints on Φ is contained in the R-neighborhood of Φ. We call such ϕ a **rank 1** element.

Since any two word metrics on Γ with respect to finite systems of generators are quasi-isometric, the notion of rank 1 does not depend on the choice of the word metric. Note that rank 1 elements in Γ have infinite order.

5.9 Theorem. *Let Γ be a group of isometries of a locally compact Hadamard space X acting cocompactly and properly discontinuously.*

Then Γ is finitely generated and $\phi \in \Gamma$ is of rank 1 if and only if ϕ is axial and one (and hence, any) axis of ϕ does not bound a flat half plane.

Proof. Since X is locally compact, Γ is finitely generated. Note that for any $x \in X$ the map $\chi : \Gamma \to X$, $\chi(\gamma) = \gamma x$ is a quasi-isometry between Γ and X. Hence, for any $A \geq 1$, $B \geq 0$ there are $A' \geq 1$, $B' \geq 0$ such that for any (A, B)-quasigeodesic $c : [a, b] \to \Gamma$ the curve $c'(\cdot) = c(\cdot)x$ is an (A', B')-quasigeodesic in X.

Now assume that $\phi \in \Gamma$ is axial and that an axis σ of ϕ does not bound a flat half plane. Choose $x = \sigma(0)$. Then $c(a)x, c(b)x \in \sigma$ and, by Theorem 5.2, $c(\cdot)x$ is contained in the $R(A', B')$-neighborhood of σ. Since χ is a quasi-isometry, $c([a, b])$ is contained in the R-neighborhood of $\{\phi^k\}$ for an appropriate R. This proves the "if" statement of the theorem. The other direction is obvious. $\square$

6. Euclidean buildings and products of trees

We assume throughout this section that (X, Γ) is a 2-dimensional orbihedron of non-positive curvature, that all links of X have diameter π and that all faces of X are flat Euclidean triangles, in particular, all edges are geodesics.

6.1 Lemma. *Let Λ be a connected graph such that the valence of each vertex is at least 3. Assume that Λ has a length structure with injectivity radius and diameter equal to π. Then:*

- (i) *Every geodesic of length $\leq \pi$ is contained in a closed geodesic of length 2π.*
- (ii) *If ξ is a vertex then any η with $d(\xi, \eta) = \pi$ is also a vertex.*
- (iii) *There is an integer $k \geq 1$ such that every edge of Λ has length π/k.*
- (iv) *If ξ and η are not vertices and $d(\xi, \eta) = \pi$ then there is a unique closed geodesic of length 2π containing ξ and η.*
- (v) *If ξ and η are vertices, $d(\xi, \eta) = \pi$ and e, f are two edges adjacent to η then there is a unique closed geodesic of length 2π containing e, f, ξ, η.*

Proof. Let σ be a geodesic of length π with ends α and ω. Whether ω is a vertex or not, there is a way to continue σ locally as a geodesic beyond ω to a point ζ such that ω and ζ lie on the same edge. It follows from our assumptions that $d(\alpha, \omega) = \pi$ and $d(\alpha, \zeta) = \pi - d(\zeta, \omega) < \pi$. Therefore, the unique shortest connection from α to ζ together with the continuation of σ form a closed geodesic of length 2π. This proves (i).

To prove (ii) let $d(\xi, \eta) = \pi$. Then, by (i), ξ and η lie on a closed geodesic γ of length 2π. Let e be an edge adjacent to ξ and not contained in γ and let ζ be a point on e different from ξ. As in the proof of (i), $d(\eta, \zeta) = \pi - d(\zeta, \xi) < \pi$. Hence, ζ lies on a geodesic arc ρ of length π from ξ to η. Since the injectivity radius of Λ is π, the arc ρ intersects γ only at ξ and η. Hence, η is a vertex. This proves (ii).

Let e, f, g be three edges adjacent to a vertex ξ. Continue g to a geodesic σ of length π and assume that $l(e)$:=length(e) <length(f). Let ζ be the point on σ with $d(\xi, \zeta) = \pi - l(e)$. Then the distance from the other end of e to ζ is π, and hence, ζ is a vertex by (ii). Let η be the point on f with $d(\eta, \xi) = l(e)$. Then $d(\eta, \zeta) = \pi$, and hence, η is a vertex. This is a contradiction since η lies in the interior of f. Hence, the lengths of any two adjacent edges are equal. This proves (iii).

To prove (iv) assume that ξ lies in an edge e with ends ξ_1, ξ_2 and η lies in an edge f with ends η_1, η_2. Since $d(\xi_i, \eta) < \pi$, there is a unique shortest connection ω_i from ξ_i to η, $i = 1, 2$. WLOG assume that $\eta_i \in \omega_i$. Since the injectivity radius of Λ is π, we have $\omega_1 \cap \omega_2 = \eta$ and $\omega_1 * e * \omega_2$ is the unique closed geodesic of length 2π containing ξ and η. This proves (iv).

To prove (v) let $\eta_e, \eta_f \neq \eta$ be the other ends of e and f, respectively. Then $0 \leq d(\eta_e, \xi), d(\eta_f, \xi) < \pi$, and hence, there are unique shortest connections ω_e, ω_f from η_e, η_f to ξ, respectively. Similarly to the proof of (ii), $e * \omega_e * \omega_f * f$ is the unique closed geodesic containing e, f, ξ. $\square$

6.2 Lemma. *Every geodesic in X is contained in a flat plane.*

Proof. It is sufficient to show that for every l every geodesic σ of length l is the middle horizontal line of a flat $l \times l$ square. By subdividing the faces of X if necessary, we may assume that l is greater than the maximal length of an edge and that all angles are $\leq \pi/2$. Set $A = \{a \geq 0 : \sigma$ is the horizontal middle line of a flat $l \times a$ rectangle$\}$. Let $a_0 = \sup A$. By the local compactness of X, the set A is closed and there is a flat $l \times a_0$ rectangle for which σ is the middle horizontal line. We will show now that A is open in $[0, \infty)$.

Let σ be the middle horizontal line of a flat $l \times a$ rectangle R whose top and bottom boundaries are geodesics $\sigma^{\pm} : [0, l] \to X$. We will extend R beyond σ^+ and σ^- by flat strips of width $\varepsilon > 0$. We will deal only with σ^+, the argument for σ^- is the same. Assume first that σ^+ does not contain an open subsegment of an essential edge. For any $t \in (0, l)$ the incoming $\xi(t)$ and outgoing $\eta(t)$ directions of σ^+ at $\sigma^+(t)$ lie at distance π in $S_{\sigma^+(t)}$ and are not vertices. Hence, by Lemma 6.1(iv), there is a unique closed geodesic in $S_{\sigma^+(t)}$ containing $\xi(t)$ and $\eta(t)$. It follows that σ^+ is contained in a unique flat strip S of positive width ε. The strip S extends R.

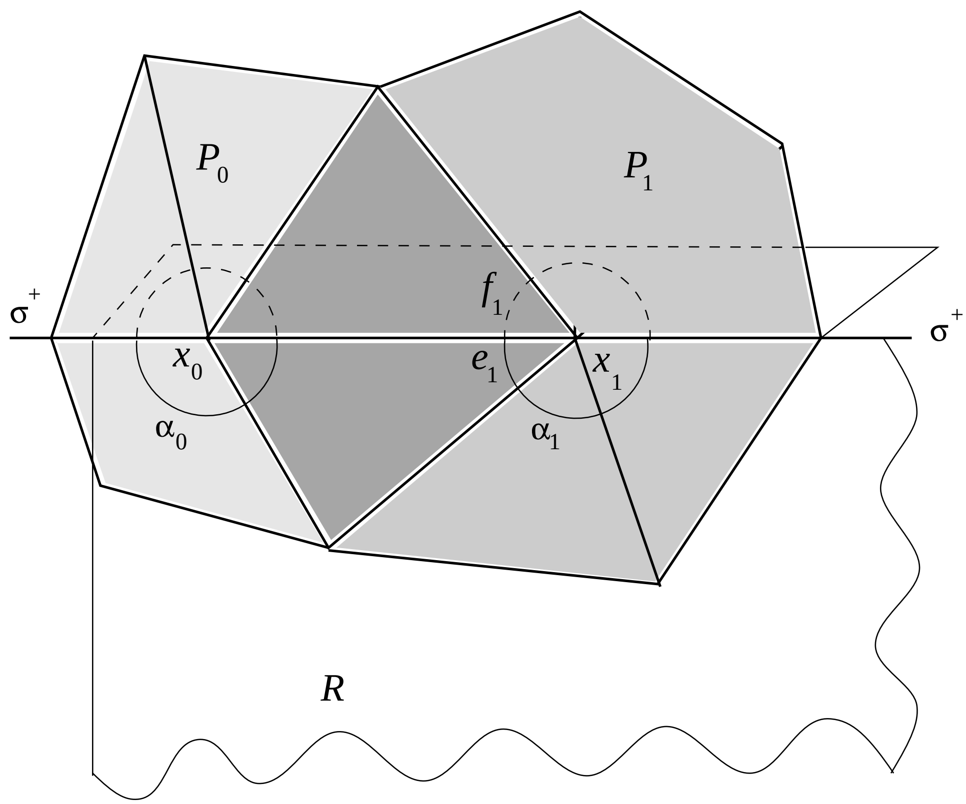

FIGURE 6

Assume now that σ^+ contains an open subsegment of an essential edge. Then, by Lemma 6.1(ii) and Lemma 6.1(v), it consists of essential edges and maybe two segments of essential edges at the ends. Let $t_0 \geq 0$ be the minimal value of the parameter for which $\sigma^+(t_0)$ is a vertex, see Figure 6. By assumption $t_0 < l$. Rectangle R is represented in the link $S_{\sigma^+(t_0)}$ by a geodesic arc α_0 of length $\pi/2$ if $t_0 = 0$ and of length π if $t_0 > 0$. In either case extend α_0 to a closed geodesic ω_0 in $S_{\sigma^+(t_0)}$. Extend R by the union P_0 of closed faces adjacent to $\sigma^+(t_0)$ which are represented by the edges forming ω_0 in $S_{\sigma^+(t_0)}$. Note that P_0 is convex since all angles are $\leq \pi/2$. Assume that t_1, $t_0 < t_1 \leq l$, is the next parameter value for which $\sigma^+(t_1)$ is a vertex. Then P_0 is represented in $S_{\sigma^+(t_1)}$ by two adjacent edges e_1, f_1 with the incoming direction $\xi(t_1)$ of σ^+ adjacent to both. One of the edges, say e_1, lies in the arc α_1 representing R in $S_{\sigma^+(t_1)}$. Note that α_1 has length π if $t_1 < l$ and $\pi/2$ if $t_1 = l$. By Lemma 6.1(v), there is a closed geodesic ω_1 of length 2π which contains f_1 and α_1 and is unique if $t_1 < l$. Let P_1 be the union of closed faces adjacent to $\sigma^+(t_1)$ which are represented by the edges forming ω_1 in $S_{\sigma^+(t_1)}$. Then P_1 is convex and $P_0 \cap P_1$ consists of the two faces represented by e_1 and f_1 in $S_{\sigma^+(t_1)}$. Therefore, we can extend $R \cup P_0$ by P_1 to a bigger flat surface. We repeat this process until we construct a flat surface containing R and a flat strip of positive width which extends it. □

6.3 Lemma. *Let Λ be a connected graph with a length structure of injectivity radius and diameter equal to π. Assume that the length of each edge in Λ is $\pi/2$.*

Then Λ is a complete bipartite graph.

Proof. Let ξ, η be two vertices in Λ with $d(\xi, \eta) = \pi$. Let U and W be the sets of vertices that lie at distance $\pi/2$ and π from ξ, respectively. Clearly there is a vertex $\zeta \in U$ such that $d(\xi, \zeta) = d(\zeta, \eta) = \pi/2$. Let $\zeta' \in U$. If $d(\zeta', \eta) \neq \pi/2$ then $d(\zeta', \eta) = \pi$. Note that $d(\zeta', \zeta) = \pi$. Hence the distance from any point between η and ζ to ζ' is greater than π. Contradiction. Hence $d(\zeta', \eta) = \pi/2$. □

6.4 Lemma. *Let X be a simply connected, locally finite complex of nonpositive curvature such that all maximal faces of X are flat rectangles and all links have diameter π.*

Then X is a product of two trees.

Proof. Since X has nonpositive curvature, every link satisfies the assumptions of Lemma 6.3, and hence is a complete bipartite graph. Fix a vertex $v_0 \in V_X$ and declare it a marked vertex. Choose a vertex $\xi_0 \in S_{v_0}$ and mark it "horizontal". Mark "vertical" all vertices η with distance $\pi/2$ to ξ_0 and mark "horizontal" all vertices ξ in S_{v_0} with distance π to ξ_0. Mark "horizontal" all edges adjacent to v_0 which are represented by horizontal vertices in S_{v_0} and mark "vertical" all other edges adjacent to v_0. Let w be any vertex of X connected to a marked vertex v by an edge e. Assign to the vertex ξ_w representing e in S_w the marking of e. Now mark accordingly the rest of the vertices in S_v, that is the vertices with distance π from ξ_w get the same marking as ξ_w and the vertices with distance $\pi/2$ get the opposite marking. We claim that this process can be used to mark consistently all vertices in the links of X and all edges in X. Since X is simply connected, it is sufficient to check that no contradictions arise for one face of X. Let u be a vertex of a maximal face F, let v, w be the vertices of F connected to u by edges e, f, and let u' be the diagonally opposite vertex of the rectangle F. Assume that all vertices in the link S_u have horizontal or vertical markings. Note that $\angle(e, f) = \pi/2$, and hence, e and f have different markings. WLOG assume that e is horizontal and f is vertical. Let g be the edge connecting v to u' and let h be the edge connecting w to u'. Then $\angle(e, g) = \angle(f, h) = \pi/2$, and hence, g is vertical and h is horizontal. Therefore, the markings for $S_{u'}$, obtained by moving through $e * g$ and through $f * h$, coincide.

Hence, every edge of X is marked either "horizontal" or "vertical". When two edges e, f are adjacent, they have the same markings if $\angle(e, f) = \pi$ and different markings if

$\angle(e, f) = \pi/2$. Now let w be any vertex. Denote by T_h the connected component of the union of horizontal edges which contains w and denote by T_v the connected component of the union of vertical edges which contains w. It is clear now that $X = T_h \times T_v$. □

6.5 Theorem. *Let (X, Γ) be a compact 2-dimensional orbispace without boundary and of nonpositive curvature. Assume that all links of X have diameter π, that all faces of X are Euclidean triangles and that all edges are geodesics.*

Then either all angles between essential edges of X are $\pi/2$ and π and X is the product of two trees, or at least one angle is π/k, $k \geq 3$, and X is a thick Euclidean building of type A_2, B_2, or G_2.

Proof. By Lemma 6.2, X is the union of embedded flat planes. Let F be such a plane. It follows from 6.1(ii) that any line σ in F containing an essential edge of X is the union of essential edges.

Suppose first that F does not contain essential vertices of X. Then, by what we said above, the union of essential edges in F is a set of parallel lines in F (an intersection would produce an essential vertex). If F does not contain an essential edge then $X = F$ since a fundamental domain of Γ has finite radius. If $X \neq F$, let $x \notin F$ and let y be the point in F closest to x. Then there is a line σ of essential edges in F through y, such that the geodesic γ from x to y is perpendicular to σ at y (recall that $\operatorname{diam} S_y = \pi$). Hence, a ray γ' in F from y and perpendicular to σ is a geodesic continuation of γ, and $\gamma * \gamma'$ is contained in a flat plane F'. By our assumption on F we have $F \cap F' = H$, where H is the half plane in F determined by σ and γ'. It follows that all essential edges in F' are parallel to σ in F'. We can see now that X is the product of an (essential) tree with a line (in the direction of σ).

Suppose now that F contains an essential vertex v. If $\alpha = \alpha(v)$ is the common length of edges in S_v, see Lemma 6.1, then $\alpha = \pi/m$ for some $m \geq 2$. Hence there are m lines of essential edges in F passing through v such that the angle between consecutive lines is π/m. These lines cut out m triangular surfaces from F. If F does not contain another essential vertex, then a fundamental domain of Γ lies completely inside one of these triangular surfaces, a contradiction. Hence there is another essential vertex in F. Since the angles at essential vertices are π/n, $n \geq 2$, it follows that the maximal faces of X in F are either rectangles or triangles.

Suppose that a maximal face Δ in F is a triangle and let e be an edge of Δ. By Lemma 6.1(iii), the maximal face Δ' of X in F opposite to Δ along e has interior angles at the ends of e equal to the interior angles of Δ. Hence Δ' is the reflection of Δ along e. It follows that the maximal faces in F are all isometric and that the tesselation of F by them is of type A_2, B_2 or G_2.

Now let e' be any essential edge in F with ends v', w' and $\Delta' \supset e'$ be any maximal face not lying in F. Take a segment ω in Δ' which is parallel to e' and close to it. Extend ω to a geodesic σ. By Lemma 6.2, there is a flat plane F' containing σ, and hence, containing Δ'. The angles α, β at v', w' in F' are the same as in F. Hence, Δ' is equal to the triangles in F. Therefore, all maximal faces of X are equal triangles. Note that any flat plane F in X is partitioned into triangles that are maximal faces of X and this partition is invariant under the reflections with respect to all essential edges in F.

We claim now that X is a thick Euclidean building whose apartments are flat planes and chambers are maximal faces of X. We must verify the following properties.

1) X is the union of flat planes. That is so since every geodesic lies in a flat plane.

2) If the intersection $F_1 \cap F_2$ of two planes contains a maximal face Δ, then there is a unique Coxeter isomorphism between F_1 and F_2 fixing $F_1 \cap F_2$. This follows from the fact that $F_1 \cap F_2$ is convex and the position of any triangle in a flat plane F_i uniquely determines the positions of all other triangles.

3) For any two maximal faces Δ_1, Δ_2 of X there is a flat plane F containing both. To see this connect the centers of Δ_1 and Δ_2 by a geodesic σ. By Lemma 6.2, there is a flat plane which contains σ, and hence, contains both Δ_1 and Δ_2.

Assume now that a maximal face $\Delta \subset F$ is a rectangle. Then the argument above shows that all maximal faces of X are rectangles and the length of each edge in each essential link is $\pi/2$. Then, by Lemma 6.3, each link of X is a complete bipartite graph, and by Lemma 6.4, X is the product of two trees. $\square$

7. Rank 1 orbihedra

In this section we will consider the situation complementary to that of Section 6 and will show that Γ contains a rank 1 isometry. We start with several lemmas that allow us to reduce step by step the class of spaces X. After Proposition 7.7 we are reduced to the situation when all edges of X are geodesics, all faces are flat triangles and the angles between essential edges in all links are rational. To handle this case we introduce a **parallel dihedral structure** in X and prove in Proposition 14 that, if there is a link in X of diameter $> \pi$, then Γ has rank 1.

Let (X, Γ) be a compact 2-dimensional boundaryless orbihedron with a piecewise smooth metric of nonpositive curvature. We fix a Γ-invariant triangulation on X such that the metric d on X is induced by a piecewise smooth Riemannian metric on this triangulation.

7.1 Lemma. *Assume that*

(i) *there is a point x_0 in an open face F of X such that the curvature of F at x_0 is negative, or*

(ii) *there is a point x_0 in an open edge e of X such that the sum of the geodesic curvatures of e in two adjacent faces F_1, F_2 is negative at x_0.*

Then there is a Γ-closed geodesic σ such that in Case (i) σ passes through a point $x \in F$ and the curvature at x is negative, and in Case (ii) σ passes from F_1 to F_2 through a point $x \in e$ and the sum of the geodesic curvatures of e at x with respect to F_1 and F_2 is negative. In both cases σ is hyperbolic.

Proof. In either case there is a geodesic ω such that $\omega(0) = x_0$, ω does not bound a flat strip and does not pass through vertices. By the Poincaré recurrence theorem (see Corollary 3.5), there are geodesics $\omega_n \to \omega$, isometries $\phi_n \in \Gamma$ and real numbers $t_n \to \infty$ such that $\phi_n^{-1}(g^{t_n}(\omega_n)) \to \omega$. It follows that $\phi_n x_0 \to \omega(\infty)$ and $\phi_n^{-1} x_0 \to \omega(-\infty)$ as $n \to \infty$. By Lemma 2.7, if n is large enough then ϕ_n has an axis σ_n and $\sigma_n(\pm\infty) \to \omega(\pm\infty)$ as $n \to \infty$. Since ω does not bound a flat strip, $\sigma_n \to \omega$ as $n \to \infty$. Hence, for n large enough the axis σ_n passes through a point x as claimed. $\square$

7.2 Lemma. *Assume that there is a vertex v_0 of X whose link S_{v_0} has the property that for any point $\xi \in S_{v_0}$ the set $\{\eta \in S_{v_0} : d(\xi, \eta) > \pi\}$ is not empty. This happens, in particular, when S_{v_0} is not connected.*

Then there is a Γ-closed geodesic σ passing through v_0 and making an angle $> \pi$ at v_0.

Proof. By the compactness of the link S_{v_0}, there is $\delta > 0$ such that for any $\xi \in S_{v_0}$ there is a point $\eta \in S_{v_0}$ with $d(\xi, \eta) > \pi + \delta$. Let ξ_0 be any point in S_{v_0} and let σ_0 be any geodesic in X such that $\sigma_0(0) = v_0$ and the outgoing direction of σ_0 in S_{v_0} is ξ_0. Denote by R the diameter of a fundamental domain for Γ. Then there is an isometry $\psi_0 \in \Gamma$ such that $d(\psi_0 v_0, \sigma(4R/\delta + R)) < R$. Let ω_0 be the geodesic connecting v_0 to $v_1 = \psi_0 v_0$. Then the outgoing direction η_0 of ω_0 at v_0 satisfies $d(\eta_0, \xi_0) < \delta/4$. Let ζ_0 be the incoming direction of ω_0 at v_1 and let $\xi_1 \in S_{v_1}$ be such that $d(\xi_1, \zeta_0) > \pi + \delta$. Let σ_1 be a geodesic such that $\sigma_1(0) = v_1$ and the outgoing direction of σ_1 at v_1 is ξ_1. There is an isometry $\psi_1 \in \Gamma$ such that $d(\psi_1 v_1, \sigma(4R/\delta + R)) < R$. Let ω_1 be the geodesic connecting v_1 to $v_2 = \psi_1 v_1$. Then the outgoing direction η_1 of ω_1 at v_1 satisfies $d(\eta_1, \xi_1) < \delta/4$. Set $\phi_2 = \psi_1 \psi_0$. Proceed in this manner to construct isometries ψ_n, $\phi_n = \psi_{n-1} \cdots \psi_0$, and geodesics ω_n such that the distance in $S_{\phi_n v_0}$ between the incoming direction ζ_{n-1} of ω_{n-1} and the outgoing direction η_n of ω_n is at least $\pi + 3\delta/4$. The last inequality implies that the concatenation of the geodesics ω_n is a geodesic in X. By the compactness of the link S_{v_0}, there are two integers m, n, $0 \le m < n$, such that $d(\phi_n^{-1}\eta_n, \phi_m^{-1}\eta_m) < \delta/4$. Set $\psi = \phi_n \phi_m^{-1}$ and $\sigma = \phi_m^{-1}(\omega_m * \omega_{m+1} * \cdots * \omega_{n-1})$. Then the concatenation ω of the geodesic segments $\psi^k(\sigma)$, $k \in \mathbb{Z}$, makes an angle $> \pi$ at v_0 and is an axis of ψ. $\square$

7.3 Lemma. *Let v be a vertex in X and suppose that $\xi, \eta \in S_v$ are such that $d_v(\xi, \eta) = \pi$.*

Then for any $\varepsilon > 0$ there are $\xi', \eta' \in S_v$, an isometry $\phi \in \Gamma$ and a geodesic ω connecting v to ϕv such that the outgoing direction of ω at v is ξ', the incoming direction of ω at ϕv is $\phi\eta'$ and $d_v(\xi', \xi), d_v(\eta', \eta) < \varepsilon$.

Proof. We subdivide the faces of X, if necessary, by the geodesic segments e and f starting at v in the directions ξ and η, respectively, and assume WLOG that e and f are edges of X. Let θ be a shortest connection from ξ to η in S_v. Then the union of the faces of X represented by the edges in S_v forming θ is a polygon P with angle π at v between e and f. By subdividing further, if necessary, we may assume that P is convex. Let v_e and v_f be the other ends of e and f and let $e' \ne e$ and $f' \ne f$ be the other edges of P adjacent to v_e and v_f, respectively. Fix $\delta > 0$ and let $\tilde{e}'$ and $\tilde{f}'$ be the subsegments of e' and f' of length δ containing v_e and v_f, respectively. Let G be the set of geodesics σ which do not pass through vertices, contain segments connecting points from $\tilde{e}'$ to points from $\tilde{f}'$ and such that $\sigma(0) \in P$. Then G has positive Liouville measure. Hence, by the Poincaré recurrence theorem (see Corollary 3.5), there is a geodesic $\sigma \in G$, an isometry $\phi \in \Gamma$ and $T > 0$, which can be chosen arbitrarily large, such that $\phi^{-1}(g^T(\sigma)) \in G$. For a large enough T and small enough δ the geodesic ω connecting v to ϕv satisfies the requirements of the lemma. $\square$

7.4 Lemma. *Assume that there is a vertex v in X whose link has the following property: there exist points $\xi_i, \eta_i \in S_v$, $i = 1, 2, ..., n$, such that $d(\xi_i, \eta_i) = \pi$, $d(\eta_i, \xi_{i+1}) \ge \pi$, $i = 1, 2, ..., n-1$, and $d(\eta_n, \xi_1) > \pi$.*

Then there is a Γ-closed geodesic σ passing through v and making an angle $> \pi$ at v.

Proof. Let $d(\eta_n, \xi_1) = \pi + \delta$ with $\delta > 0$. Fix any positive $\varepsilon < \delta/(2n)$ and use Lemma 7.3 to construct geodesics ω_i and isometries ϕ_i. Set $\psi_i = \phi_i ... \phi_2 \phi_1$ and $\omega = \omega_1 * \psi_1 \omega_2 * ... * \psi_{n-1}\omega_n$. Then ω consists of geodesic segments with angles $> \pi - 2\varepsilon$ at the $n-1$ break points, its starting direction is at distance $< \varepsilon$ from ξ_1 and its ending direction at $\psi_n(v)$ is at distance $< \varepsilon$ from $\psi_n(\eta_n)$. By Lemma 2.5, the incoming direction of the geodesic σ from v to $\psi_n(v)$ at $\psi_n(v)$ and the image of its outgoing direction under ψ_n lie at distance $> \pi$ in $S_{\psi_n(v)}$. Hence, the geodesic

$$... * \psi_n^{-k}\sigma * ... * \psi_n^{-1}\sigma * \sigma * \psi_n \sigma * ... * \psi_n^k \sigma * ...$$

is an axis of ψ_n and satisfies the requirements. $\square$

The assumption $\operatorname{diam} S_v > \pi$ is not sufficient for the existence of a finite sequence of pairs of points ξ_i, η_i as in Lemma 7.4 – the 1-skeleton of a tetrahedron with all edges of length $2\pi/3$ is a counterexample. However, as an immediate consequence of Lemma 7.4 we have

7.5 Corollary. *Let v be a vertex in X such that $\operatorname{diam} S_v > \pi$ and there is a closed geodesic in S_v whose length is an irrational multiple of π.*

Then there is a hyperbolic axial isometry $\psi \in \Gamma$. □

7.6 Lemma. *Let v be a vertex in X such that the link S_v contains a simple arc ω of length $l > \pi$ whose end points $\xi \neq \eta$ are essential vertices and whose interior does not contain essential vertices.*

Then there is a Γ-closed geodesic σ passing through v and making an angle $> \pi$ at v.

Proof. If $l > 2\pi$, the statement follows from Lemma 7.2. We assume that $l < 2\pi$, the argument for the case $l = 2\pi$ is similar. Let ζ be the midpoint of ω. Since ξ is essential, there are at least two ways of extending the subarc from ζ to ξ beyond ξ to arcs $\omega_{\xi 1}, \omega_{\xi 2}$ of length π. Similarly, there are at least two extensions $\omega_{\eta 1}, \omega_{\eta 2}$ of $[\zeta, \eta] \subset \omega$ beyond η to arcs of length π. Recall that $l > \pi$, the injectivity radius is π and there are no essential vertices in the interior of ω. Hence there exist indices i, j such that $\omega_{\xi i}$ does not intersect any arc of length $\pi - l/2$ starting at η and $\omega_{\eta j}$ does not intersect any arc of length $\pi - l/2$ starting at ξ. Therefore, ω can be extended to a simple arc ω' in S_v of length $2(\pi + \varepsilon)$, $\varepsilon > 0$, which contains $\omega_{\xi i}$ and $\omega_{\eta j}$, does not intersect other arcs from ξ and η of length $\pi - l/2$, and for which ζ is the midpoint. Let α and β denote the ends of ω'. Consider the following points on ω: ξ_0 lies on $\omega_{\xi i}$ at distance ε from ζ, $\eta_0 = \alpha$, ξ_1 lies on $\omega_{\eta j}$ at distance ε from ζ, $\eta_1 = \beta$. By construction, $d_v(\xi_0, \eta_1), d_v(\xi_1, \eta_0) > \pi$. Hence, Lemma 7.4 applies and the lemma follows. □

7.7 Proposition. *Let Λ be a finite graph with a length structure of injectivity radius 1. Assume that every vertex is adjacent to at least 3 edges and that the length of every closed geodesic in Λ is rational.*

Then the length of every edge is rational.

Proof. If an edge connects a vertex to itself then it is a closed geodesic and its length is rational by assumption. To treat other cases we need the following auxiliary statement.

7.8 Lemma. *If f is an oriented edge with different ends then there is a geodesic loop c in Λ starting with f and ending with f^{-1}.*

Proof. Let e connect v to $w \neq v$. By assumption, $\Lambda \setminus f$ is a nonempty graph with every vertex adjacent to at least 2 edges. Therefore, $\Lambda \setminus f$ contains a geodesic loop c' at w. Now let $c = f * c' * f^{-1}$. □

We continue now with the proof of Proposition 7.7. Let e be an edge connecting x to $y \neq x$. If there is an edge $f \neq e$ starting and ending at x then $f := c_x$ is a closed geodesic at x. Otherwise there are 2 oriented edges f_1, f_2 starting but not ending at x. By Lemma 7.8, there are geodesic loops c_1, c_2 for f_1, f_2 such that $c_x := c_1 * c_2$ is a closed geodesic at x. We construct c_y in a similar way and note that $c = c_x * e * c_y * e^{-1}$ is a closed geodesic and $2\,\mathrm{length}(e) = \mathrm{length}(c) - \mathrm{length}(c_x) - \mathrm{length}(c_y)$. This finishes the proof of Proposition 7.7 □

By Proposition 2.11, Lemma 7.1 and Lemma 7.2 we can assume that all edges of X are geodesics, that all faces are Euclidean triangles and that all links are connected. By Corollary 7.5, Lemma 7.6 and Proposition 7.7 we can assume that all angles between adjacent essential edges are rational multiples of π and $\leq \pi$.

7.9 Definition. A **parallel dihedral structure** D of order q in X is a family of subsets $D_x \subset S_x$, $x \in X$, such that

(i) for any $x \in X$ and $\xi \in D_x$ we have

$$D_x = \{\eta \in S_x \,:\, d(\xi,\eta) = \frac{k\pi}{q} \text{ for some integer } k\};$$

(ii) if $x, y \in X$ belong to the same open face $\mathring{F}$ then D_x and D_y are parallel in $\mathring{F}$;
(iii) if $x \in X$ lies in a closed face F and $\xi \in D_x$ belongs to $T_x F$ then for any y in the interior $\mathring{F}$ there is $\eta \in D_y$ parallel to ξ in F;
(iv) if $x \in X$ lies in an essential edge e then the point $\xi \in S_x$ representing e lies in D_x.

Clearly (m,n)-complexes with their canonical piecewise flat metric (see Section 1) are examples of complexes with a parallel dihedral structure.

7.10 Remark. If X has a parallel dihedral structure of order q then

(i) the length of any closed geodesic in any link of X is an integer multiple of π/q,
(ii) if x_i lies in an open face $\mathring{F}_i$, $i = 1, 2$, and the faces F_1, F_2 are adjacent by an edge then $D_{x_1} \parallel D_{x_2}$ in the union of the faces.

Recall that a maximal face of X is a connected component of the union of all open faces, inessential edges and interior vertices (see Section 2).

7.11 Lemma. *If all angles between adjacent essential edges of X are $\leq \pi$, then every maximal face of X is locally convex.* $\square$

7.12 Proposition. *If the angles between the essential edges of X are rational multiples of π and $\leq \pi$, then X has a parallel dihedral structure.*

Proof. The assumptions do not immediately imply the existence of a parallel dihedral structure because the union E' of closed essential edges of X may be disconnected.

If $E' = \emptyset$ then X has a parallel dihedral structure by the well known classification of planar lattices. If $E' \neq \emptyset$ then, by Lemmas 7.11 and 2.4, the maximal faces of X are embedded polygonal subsets in the plane. Since Γ acts cocompactly, maximal faces of X do not contain arbitrarily large disks. Therefore, we have the following possibilities for a maximal face F: 1) a convex polygon, 2) an infinite flat strip, 3) two parallel rays whose ends are connected by a finite polygonal line, see Figure 7. Let q be the common denominator of all angles between adjacent essential edges. If $x \in X$ is a point lying on an essential edge e then let $D_x \subset S_x$ consist of the point ξ representing e and all points $\eta \in S_x$ for which $d_x(\eta,\xi)$ is an integer multiple of π/q. For any other $y \in X$ lying in the interior $\mathring{F}$ of a maximal face F choose any $x \in \partial F$ and define D_y as the parallel translation of D_x from x to y in F. In Cases 1) and 3) all essential edges forming ∂F belong to the same connected component of E', and hence, the set D_y does not depend on the choice of $x \in \partial F$. In Case 2) there are two connected components of E' but they are parallel, and D_y also does not depend on where x lies in ∂F. It is easy to see that this defines a parallel dihedral structure on X. $\square$

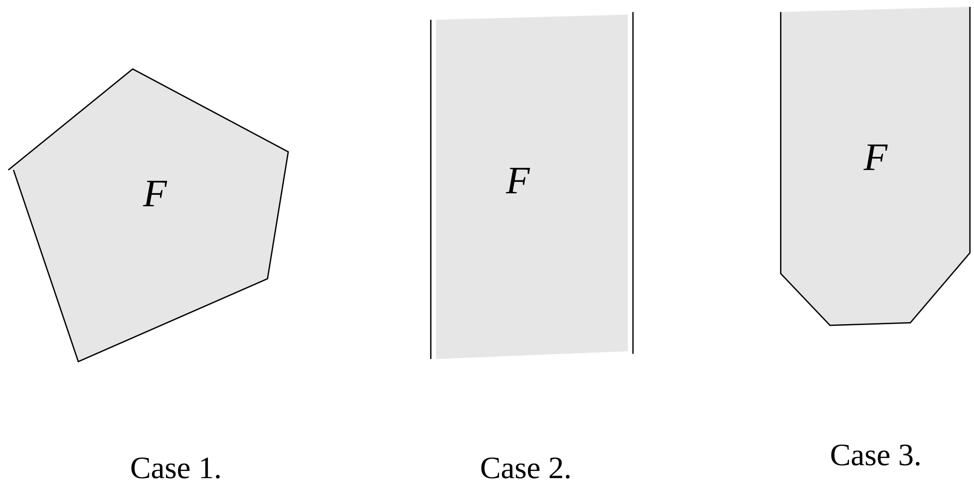

FIGURE 7

7.13 Lemma. *Let X have a parallel dihedral structure D and let ϕ be an axial isometry with an axis σ which bounds a flat half plane and whose direction does not belong to D.*

Then the set P of geodesics parallel to σ is a plane and Γ contains a subgroup acting cocompactly on P.

Proof. Since the direction of σ is not in D, the set P is the product of a line (in the direction of σ) and an interval. Since σ bounds a flat half plane, the interval is infinite. Assume that P is not a plane. Then it is exactly a flat half plane with boundary σ' invariant under ϕ.

Let F be a fundamental domain of Γ, set $\sigma'(0) = x_0$ and let x_n be the point in P that lies on the perpendicular to σ' through x_0 at distance $2n \times \operatorname{diam} F$ from x_0. Let σ_n be the geodesic passing through x_n and parallel to σ'. Note that σ_n is an axis of ϕ. There is $\psi_n \in \Gamma$ such that $y_n = \psi_n x_n \in F$. The geodesic $\psi_n(\sigma_n)$ passes through y_n, and hence, through F and is an axis of $\psi_n \phi \psi_n^{-1}$. Observe that the displacement of y_n by $\psi_n \phi \psi_n^{-1}$ is equal to the displacement of x_0 by ϕ and all y_n's lie in F. Hence, by the discreteness of Γ there are infinitely many pairs $m \neq n$ such that $\psi_n \phi \psi_n^{-1} = \psi_m \phi \psi_m^{-1}$. Hence ϕ commutes with $\psi = \psi_m^{-1} \psi_n$. Note that $\psi \neq id$ by the choice of x_n. Since ψ commutes with ϕ, it leaves invariant the set of axes of ϕ. By composing ψ with itself, if necessary, we may assume that ψ preserves orientation in P. If $m > n$ then ψ moves x_n away from σ' in P, and hence, moves x_0 away from σ' in P. This is a contradiction. Hence, P is a plane. The same argument implies that the group generated by ϕ and ψ acts cocompactly on P. □

7.14 Proposition. *Assume that X has a parallel dihedral structure D of order q and that there is a vertex v in X such that S_v has diameter $> \pi$.*

Then there is an axial isometry $\psi \in \Gamma$ with an axis which does not bound a flat half plane, and hence, Γ is of rank 1.

Proof. By doubling D if necessary we may assume that q is even. Suppose that all axes of the isometries from Γ bound flat half planes. If there is no essential edge adjacent to v then S_v is a circle and, by Lemma 7.2, there is an axial isometry in Γ with a hyperbolic axis. Hence, we may assume that there is an essential edge e adjacent to v. Choose two

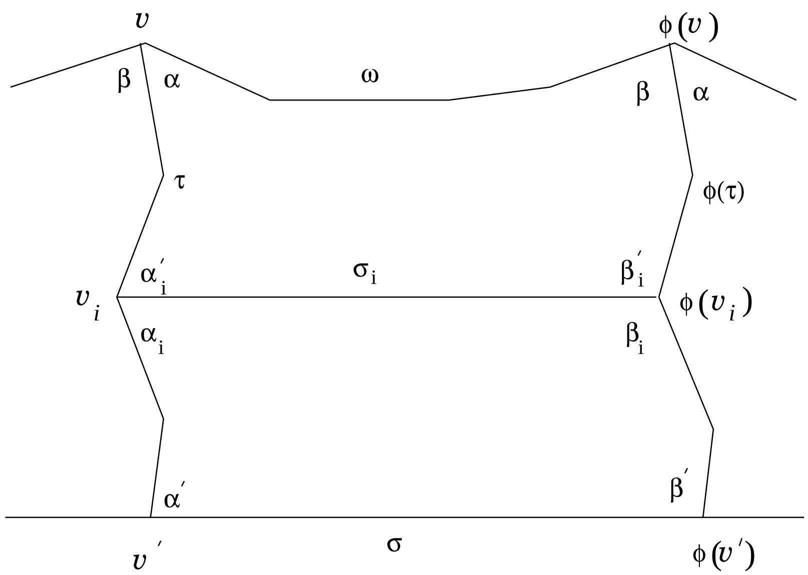

FIGURE 8

points $\xi, \eta \in S_v$ such that

(i) ξ and η lie on a minimal geodesic $\gamma \subset S_v$ of length $> \pi$,
(ii) $d(\xi, \eta) = \pi$,
(iii) $d(\xi, D_v) = d(\eta, D_v) = \dfrac{\pi}{2q}$,
(iv) the balls $B(\xi, \dfrac{\pi}{4q})$ and $B(\eta, \dfrac{\pi}{4q})$ centered at ξ and η are contained in γ.

By construction, we have:

$$\text{if } \xi' \in B(\xi, \frac{\pi}{4q}),\ \eta' \in B(\eta, \frac{\pi}{4q}) \text{ then there is no closed geodesic in } S_v \text{ of length } 2\pi \text{ containing } \xi' \text{ and } \eta'. \tag{7.15}$$

By Lemma 7.3, for any $\varepsilon > 0$ there are $\xi', \eta' \in S_v$, an isometry $\phi \in \Gamma$ and a geodesic ω connecting v to ϕv such that the outgoing direction of ω at v is ξ', the incoming direction of ω at ϕv is $\phi\eta'$ and $d_v(\xi', \xi), d_v(\eta', \eta) < \varepsilon$. By passing if necessary to the geodesic connecting the ends of $\omega * \omega$, we may assume that the isometry ϕ corresponding to ω is a square. Let σ be an axis of ϕ and let τ be the shortest connection from v to σ. Set $\alpha = \angle_v(\tau, \omega)$, $\beta = \angle_{\phi(v)}(\phi(\tau), \omega)$ and denote by α' and β' the angles formed by τ, σ and $\phi(\tau), \sigma$, respectively. Obviously, $\alpha', \beta' \geq \pi/2$. By Corollary 2.3, $\alpha + \alpha' \leq \pi$ and $\beta + \beta' \leq \pi$ since $d(v, \sigma) = d(\phi(v), \sigma)$. Note that $\alpha + \beta \geq \pi - 2\varepsilon$ by the choice of ω. Hence

$$\frac{\pi}{2} - 2\varepsilon \leq \alpha, \beta \leq \frac{\pi}{2} \leq \alpha', \beta' \leq \frac{\pi}{2} + 2\varepsilon\,.$$

We are not using it but actually $\alpha' + \beta' = \pi$, and hence, $\alpha' = \beta' = \pi/2$. Let $v_1, v_2, ..., v_n$ be the vertices lying in the interior of τ in consecutive order and let v' be the intersection point of τ and σ. Denote by σ_i the geodesic connecting v_i to $\phi(v_i)$ and let $\alpha_i, \alpha_i', \beta_i, \beta_i'$, $i = 1, ..., n$, be the angles indicated in Figure 8. Since τ and $\phi(\tau)$ are geodesics, we have

$\alpha_i + \alpha'_i \geq \pi$, $\beta_i + \beta'_i \geq \pi$, $i = 1, ..., n$. By Lemma 2.5, all these angles are between $\pi/2 - 2\varepsilon$ and $\pi/2 + 2\varepsilon$. Let δ_i denote the defect of τ at v_i, that is $\delta_i = d_i - \pi \geq 0$, where d_i is the distance in S_{v_i} between the incoming and outgoing directions of τ. Note that $\alpha_i, \alpha'_i < \pi$, and hence, α_i, α'_i realize the distance from these directions to the direction of σ_i at v_i. Therefore, $\alpha_i + \alpha'_i \geq \pi + \delta_i$. Similarly, $\beta_i + \beta'_i \geq \pi + \delta_i$. Since the sum of the angles of any geodesic quadrangle is at most 2π, we have:

$$
\begin{aligned}
2(n+1)\pi &\geq \alpha' + \beta' + \alpha_1 + \beta_1 + \sum_{i=2}^{n} (\alpha'_{i-1} + \beta'_{i-1} + \alpha_i + \beta_i) + \alpha'_n + \beta'_n + \alpha + \beta \\
&\geq \sum_{i=1}^{n} (\alpha_i + \alpha'_i + \beta_i + \beta'_i) + \alpha' + \beta' + \alpha + \beta \geq 2n\pi + 2\sum_{i=1}^{n} \delta_i + 2\pi - 4\varepsilon \,.
\end{aligned}
$$

Hence, $\sum_{i=1}^{n} \delta_i < 2\varepsilon$.

Let θ^+ denote the outgoing direction of τ at v. Then $d_v(\theta^+, D_v) \geq \dfrac{\pi}{2q} - 3\varepsilon$ since q is even and $\pi/2 - 2\varepsilon \leq \alpha \leq \pi/2$. Let θ_i^+ and θ_i^- denote the outgoing and incoming directions of τ at v_i, respectively. Then $d_{v_1}(\theta_1^-, D_{v_1}) = d_v(\theta^+, D_v) \geq \dfrac{\pi}{2q} - 3\varepsilon$, and hence, $d_{v_1}(\theta_1^+, D_{v_1}) \geq \dfrac{\pi}{2q} - 3\varepsilon - \delta_1$. Repeating this argument, we obtain for the incoming direction θ^- of τ at v'

$$
d_{v'}(\theta^-, D_{v'}) = d_{v_n}(\theta_n^+, D_{v_n}) \geq \frac{\pi}{2q} - 3\varepsilon - \sum_{i=1}^{n} \delta_i \geq \frac{\pi}{2q} - 5\varepsilon \,.
$$

Let ζ be the direction of σ at v'. Since $\pi/2 \leq \alpha' \leq \pi/2 + 2\varepsilon$, we have

$$
d_{v'}(\zeta, D_{v'}) \geq \frac{\pi}{2q} - 7\varepsilon \,. \tag{7.16}
$$

Therefore, if ε is small enough then $\zeta \notin D_{v'}$.

By our assumption, σ bounds a flat half plane. By Lemma 7.13, the set P of geodesics parallel to σ is a plane and, since ϕ a square, it acts as a translation in P in the direction of σ. WLOG assume that τ is the shortest connection from v to P. Recall that $\alpha' \leq \pi/2 + 2\varepsilon < \pi$. Hence, τ and σ locally span a unique flat sector S in X with angle α' at the apex v'. Since the direction of σ is not in D, there is a subsector S' of S containing σ and lying in P. By (7.16), the angle of S' at v' is at least $\dfrac{\pi}{2q} - 7\varepsilon$. It follows that the angle between τ and P is at most

$$
\alpha' - \left(\frac{\pi}{2q} - 7\varepsilon\right) \leq \frac{\pi}{2} - \frac{\pi}{2q} + 9\varepsilon \,.
$$

For a small enough ε the right hand side is less than $\pi/2$ which contradicts the fact that τ is the shortest connection from v to P. Hence, ω lies in P and is parallel to σ. This contradicts (7.15). $\square$

8. Euclidean buildings

A general reference for the following is [Bro]. Recall that a **Tits building** is a simplicial complex X which is the union of subcomplexes, called **apartments**, such that

(B0) each apartment is a Coxeter complex;
(B1) for any two simplices A, A' in X, there is an apartment containing both of them;
(B2) for any two simplices A, A' in X and apartments F, F' containing both of them, there is an isomorphism $F \to F'$ fixing A and A' pointwise.

We may take A and A' to be the empty simplex in $(B2)$, and hence any two apartments are isomorphic. In particular, all apartments have the same dimension. Simplices of maximal dimension are also called **chambers**. Axiom (B2) can be replaced by the following axiom, see [Bro, p.77]:

(B2') if F, F' are apartments with a common chamber C, then there is an isomorphism $i : F \to F'$ fixing $F \cap F'$ pointwise.

We say that X is a **Euclidean building** if its apartments are Euclidean Coxeter complexes. A Euclidean building has a canonical piecewise smooth metric d consistent with the Euclidean structure on the apartments and turning it into a Hadamard space.

Let X be a Euclidean building of dimension n, equipped with the complete system of apartments and the canonical metric d. Then a subset of X is an apartment if and only if it is convex and isometric to $\mathbb{R}^n$. For this reason, we call the apartments of X **flats**. Every geodesic of X is contained in a flat.

Let A_F be the group of automorphisms of a flat F preserving the triangulation. Then A_F preserves the metric, and hence, A_F is a Bieberbach group of rank n. The subgroup $T_F \subset A_F$ of translations is a normal and maximal abelian subgroup. It is free abelian of rank n and has finite index in A_F.

8.1 Remark. The triangulation of F is defined by a finite number k of pairwise transverse families $\mathcal{H}_1, \dots, \mathcal{H}_k$ of parallel hyperplanes in F which are called **walls**. The $(n-2)$-skeleton of F consists of the intersections of walls $H_i \cap H_j$ with $H_i \in \mathcal{H}_i$, $H_j \in \mathcal{H}_j$ and $i \neq j$. The open $(n-1)$-simplices are the complements of these intersections in the walls.

The Coxeter group $W_F \subset A_F$ of automorphisms of F generated by the reflections in the walls $H \in \mathcal{H} := \bigcup_{i=1}^k \mathcal{H}_i$ has finite index in A_F.

Fix a flat F and a translation $\tau \in T_F$. We say that a translation τ' of a flat F' is **conjugate** to τ if there is an isomorphism $i : F' \to F$ such that $\tau' = i^{-1} \circ \tau \circ i$. Thus a translation τ' of F is conjugate to τ if and only if τ' is conjugate to τ in A_F. It follows that the number of translations of a flat F' conjugate to τ is equal to the number m of elements in the conjugacy class of τ in A_F.

We say that a geodesic σ is **special** (with respect to τ) if σ does not meet the $(n-2)$-skeleton of X and if there is a flat F' containing σ and a translation τ' of F' conjugate to τ such that

$$\tau'(\sigma(t)) = \sigma(t + t_0) \text{ for all } t \in \mathbb{R},$$

where $t_0 = \|\tau'\| = \|\tau\| > 0$ is the displacement. This is independent of the flat F' containing σ : if F'' is another such flat, then $\sigma \subset F' \cap F''$. Since σ does not meet the $(n-2)$-skeleton of X, this implies that $F' \cap F''$ contains an n-simplex. Hence there is an isomorphism $i : F'' \to F'$ fixing σ pointwise and $\tau'' = i^{-1} \circ \tau' \circ i$ is a translation of F'' conjugate to τ and shifting σ as required.

If σ is special with respect to τ, if F' is a flat containing σ, and if $i : F' \to F''$ is an isomorphism to another flat F'', then $i \circ \sigma$ is also special with respect to τ.

8.2 Example. Let B be an open $(n-1)$-simplex in X, and let F be a flat containing B. Consider the system $\mathcal{H}$ of walls as in Remark 8.1. Then $B \subset H \in \mathcal{H}_i$ for some i, $1 \le i \le k$. If $H' \in \mathcal{H}_i$ is another wall, then the composition τ of the reflections in H and H' is a translation of F perpendicular to B. A unit speed geodesic σ in a flat F' is special with respect to this τ iff it does not meet the $(n-2)$-skeleton of X and if there is an isomorphism $i : F' \to F$ such that $i \circ \sigma$ intersects B perpendicularly.

8.3 Lemma. *Let σ be a special geodesic in a flat F'. Suppose that ω is a geodesic with $\sigma(0) = \omega(0)$ and $\dot\sigma(0) = \dot\omega(0)$. Then ω is also special. More precisely, if ω is contained in a flat F'', then there is an isomorphism $i : F'' \to F'$ with $i(\omega) = \sigma$.*

Proof. Since special geodesics do not meet the $(n-2)$-skeleton of X and intersect the $(n-1)$-skeleton transversally, there is an n-simplex C of X and an $\varepsilon > 0$ such that

$$\omega(t) = \sigma(t) \in C, \ 0 < t < \varepsilon.$$

Hence $C \subset F' \cap F''$ and therefore there is an isomorphism $i : F'' \to F'$ fixing $F' \cap F''$ pointwise. Then $i(\omega) = \tau$. □

8.4 Lemma. *Suppose τ' is a translation of a flat F' which is conjugate to τ. Denote by v the parallel field of unit vectors in F' in the direction of τ'. Let B be an open $(n-1)$-simplex of F' transverse to τ'.*

Then there is an open and dense subset $B(\tau') \subset B$ of full measure and an $\varepsilon_1 > 0$ such that a geodesic σ in F' with $x := \sigma(0) \in B$ and $d_x(\dot\sigma(0), v(x)) < \varepsilon_1$ is special iff $x \in B(\tau')$ and $\dot\sigma(0) = v(x)$.

Proof. If m is the number of elements in the conjugacy class of τ in A_F then there are m directions in F' which special geodesics can point in. Hence a geodesic σ approximately pointing in the direction of v can be special only if $\dot\sigma(0) = v(x)$, where $x = \sigma(0)$. If $\dot\sigma(0) = v(x)$, then σ is shifted by τ' and σ does not meet the $(n-2)$-skeleton of X if $\sigma([0, \|\tau'\|])$ does not meet the $(n-2)$-skeleton of F. □

We return to our discussion in Section 3. For a point x in an open $(n-1)$-simplex adjacent to an n-simplex C denote by $S''_xC \subset S'_xC$ the directions tangent to special geodesics. Then S''_xC contains at most m elements. Let $C_1, \dots, C_r$ be the n-simplices adjacent to an $(n-1)$-simplex B. We set

$$S''_x = \cup_{i=1}^r S''_x C_i \ \text{ and } \ V_\tau = \cup_{x \in X'} S''_x.$$

Then $V_\tau \subset V$ by the defining property of special geodesics.

There is a natural measure ν on V_τ (the conditional measure of μ, see (3.1)):

$$d\nu(v) = \cos\theta(v) dx, \tag{8.5}$$

where x is the foot point of v and dx the volume element of X' (see the beginning of Section 3).

By the definition of V_τ, we have $F(v) \subset V_\tau$ if $v \in V_\tau$, and hence the Markov chain with transition probabilities given by (3.2) restricts to a Markov chain with state space V_τ. One can check easily that the measure ν given by (8.3) is stationary and hence gives an invariant measure ν^* for the shift in the space V_τ^* of sequences $(v_n)_{n\in\mathbb{Z}}$ in V_τ.

Denote by G_τ the set of geodesics which are special with respect to τ. Then G_τ is invariant under the geodesic flow and under automorphisms of X. We may think of V_τ^* as a cross section in G_τ, where the return map (of the geodesic flow g^t) corresponds to the shift on V_τ^*. Hence ν^* defines an invariant measure ν_τ for g^t on G_τ.

Let Γ be a group of automorphisms of X which acts properly discontinuously and cocompactly. Since Γ leaves G_τ invariant, the measure ν_τ gives a finite invariant measure for the induced action of g^t on G_τ/Γ. Thus the Poincaré recurrence theorem implies the following corollary.

8.6 Corollary. *Every special geodesic σ is* **nonwandering** *mod Γ, that is, there are sequences $\sigma_n \in G_\tau$, $\phi_n \in \Gamma$ and $t_n \in \mathbb{R}$ with $\sigma_n \to \sigma$, $t_n \to +\infty$ and $\phi_n(g^{t_n}(\sigma_n)) \to \sigma$.* □

Recall that special geodesics intersect the $(n-1)$-skeleton transversally.

8.7 Lemma. *Let σ be a special geodesic intersecting an $(n-1)$-simplex B transversally at $x = \sigma(0)$.*

Then there is an $\varepsilon_2 > 0$ with the following property: if ω is a geodesic with $\omega(0) = x$ and $\dot\omega(0) = \dot\sigma(0)$, if $\omega(t)$ is in an $(n-1)$-simplex B' and if $f : B' \to B$ is an isomorphism (of simplices) with $d(f(\omega(t)), x) < \varepsilon_2$, then $f(\omega(t)) = x$.

Proof. Let F' be a flat containing σ and F'' a flat containing ω. By Lemma 8.3, ω is special and hence $B' \subset F''$. Furthermore, there is an isomorphism $i : F'' \to F'$ with $i(\omega) = \sigma$. Hence we may assume $\omega = \sigma$ and $F'' = F'$.

Now σ is shifted by a translation τ' of F' conjugate to τ. Since τ' is an automorphism of F', there are only finitely many (combinatorial) possibilities for the intersection of σ with $(n-1)$-simplices. □

8.8 Lemma. *Suppose $\bar\sigma$ is a finite segment of a special geodesic σ.*

Then there is a Γ-closed geodesic containing $\bar\sigma$.

Proof. Let F' be a flat containing σ. By reparameterizing σ and enlarging the given segment of σ if necessary, we may assume that $x = \sigma(0)$ is in an open $(n-1)$-simplex B and that $\bar\sigma = \sigma([0,T])$ with $T > 0$. By Lemmas 8.3 and 8.4, there is an $\varepsilon > 0$ with the following properties:

(1) for any special geodesic ω intersecting the balls $B_\varepsilon(\sigma(-1))$ and $B_\varepsilon(\sigma(T+1))$, the segment $\omega \cap F'$ is parallel to σ in F'.

(2) if ω is a geodesic with $\omega(0) \in F'$, $\dot\omega(0)$ parallel to $\dot\sigma(0)$ in F' and $d(x, \omega(0)) < \varepsilon$, then ω is special.

By Corollary 8.6, there are sequences $\sigma_n \in G_\tau$, $\phi_n \in \Gamma$ and $t_n \in \mathbb{R}$ such that $\sigma_n \to \sigma$, $t_n \to \infty$ and $\phi_n g^{t_n}\sigma_n \to \sigma$. Since σ does not meet the $(n-2)$-skeleton of X, we conclude that

$$\sigma_n(t), (\phi_n g^{t_n}\sigma_n)(t) \in F, \quad -1 \le t \le T+1$$

for all n sufficiently large. By a small change of the parameterization of σ_n and a small change of t_n we may assume that $\sigma_n(0)$ and $(\phi_n g^{t_n}\sigma_n)(0)$ are in B.

Now σ_n and $\phi_n g^{t_n}\sigma_n$ are special. For n large enough they intersect $B_\varepsilon(\sigma(-1)$ and $B_\varepsilon(\sigma(T+1))$, and then $\sigma_n \cap F'$ and $(\phi_n g^{t_n}\sigma_n) \cap F'$ are parallel to σ in F' by (1). In particular, $\dot\sigma_n(0)$ and $\phi_{n*}(\dot\sigma_n(t_n))$ are parallel to $\dot\sigma(0)$ in F'.

Let F_n be a flat containing σ_n. Since special geodesics do not meet the $(n-2)$-skeleton, $\sigma([0,T]) = \bar\sigma$ is in F_n and parallel to σ_n in F_n for all n sufficiently large. Let ω_n be the geodesic in F_n with $\omega(0) = x$ and $\dot\omega_n(0) = \dot\sigma(0)$. Then ω_n is parallel to σ_n in F_n and $\bar\sigma$ is contained in ω_n. Since $\phi_n g^{t_n}\sigma_n \to \sigma$ and $d(\omega_n, \sigma_n) \to 0$, we conclude that $d(\omega_n(t_n), x) \to 0$. Hence, by Lemma 8.7 $\phi_n(\omega_n(t_n)) = x$ for all n sufficiently large. Since $\phi_{n*}(\dot\sigma_n(t_n))$ is parallel to $\dot\sigma(0)$ in F' and ω_n is parallel to σ_n, we have that $\phi_{n*}(\dot\omega_n(t_n)) = \dot\sigma(0) = \dot\omega_n(0)$. Therefore,

$$\cup_{k\in\mathbb{Z}}\phi_n(\omega_n([0,t_n]))$$

is a geodesic invariant under ϕ_n and containing $\bar\sigma$. $\square$

Let Γ_F denote the stabilizer of a flat F and let

$$\Gamma'_F = \{\phi \in \Gamma_F \,:\, \phi(x) = x \text{ for all } x \in F\}.$$

Then $\Delta_F := \Gamma_F/\Gamma'_F$ is (isomorphic to) a subgroup of A_F. We say that F is Γ-**closed** if Γ_F acts cocompactly on F, that is, if Δ_F has finite index in A_F. If F is Γ-closed, then $T_F \cap \Delta_F$ has finite index in T_F and Δ_F.

8.9 Theorem. *Let K be a compact subset of a flat F in X.*
Then there is a Γ-closed flat F' containing K.

Proof. Let $\tau \in T_F$ be a translation in a direction which is not tangent to any of the walls of F. Let σ be a unit speed geodesic in F shifted by τ and not passing through the $(n-2)$-skeleton of F. Then F is the unique flat containing σ.

Consider the system of pairwise transverse families $\mathcal{H}_1, \dots, \mathcal{H}_k$ of walls in F. By a **half flat** in F we mean the part of F on one side of a wall. For each i, $1 \le i \le k$, there are half flats F_i^+ and F_i^- in F with boundaries $H_i^+, H_i^- \in \mathcal{H}_i$ such that $K \subset F_i^+ \cap F_i^-$. By our assumption on σ,

$$\sigma_i := \sigma \cap (F_i^+ \cap F_i^-)$$

is a finite segment of σ. Let $\bar\sigma$ be a finite segment of σ containing σ_i, $1 \le i \le k$, see Figure 9.

By Lemma 8.8, there is a Γ-closed geodesic σ' containing $\bar\sigma$. Let F' be a flat containing σ'. By the choice of $\bar\sigma$, $F \cap F'$ is a convex subset of F with interior. Hence the boundary of $F \cap F'$ is a union of closed $(n-1)$-simplices. It follows that $F \cap F'$ is the intersection of half flats. Note that σ_i is contained in $F \cap F'$. Hence the half flat containing $F \cap F'$, bounded by a wall H in $\mathcal{H}_i$, must contain $F_i^+ \cap F_i^-$. Therefore $K \subset F \cap F'$.

There is a unique isomorphism $j : F' \to F$ fixing $F \cap F'$. Hence, $j(\sigma') = \sigma$, and σ' is not parallel to any of the walls in F' and does not pass through the $(n-2)$-skeleton of F'. It follows that F' is the unique flat of X containing σ'. Let $\phi \in \Gamma$ shift σ'. Then $\phi(F') = F'$, by the uniqueness of F'. By passing to a finite power of ϕ, we may assume

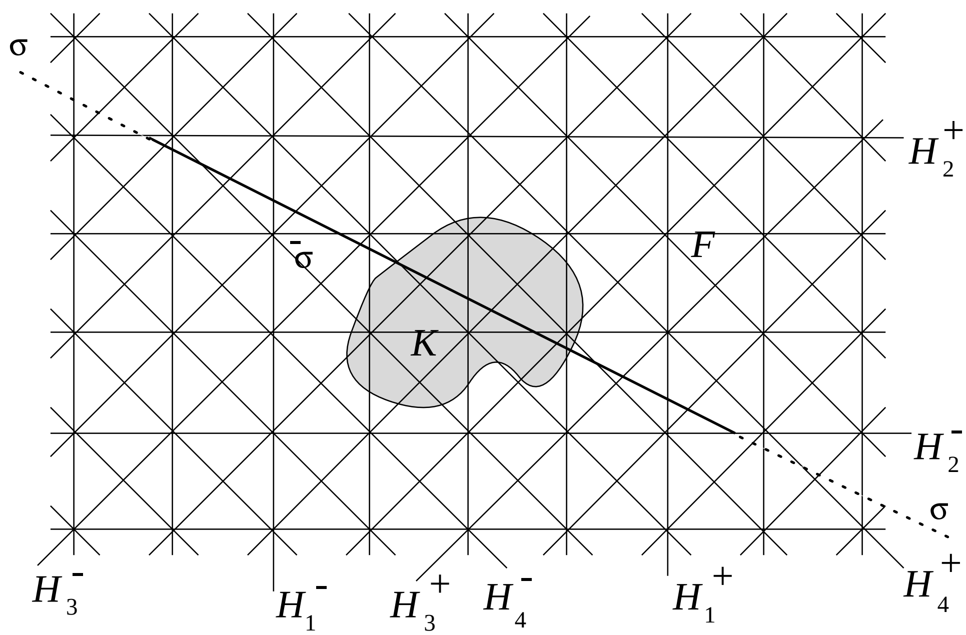

FIGURE 9

that ϕ is a translation of F'. Now the argument of Lemma 7.13 applies and finishes the proof of the theorem. □

8.10 Theorem. *Let X be a Euclidean building and Γ a group of automorphisms of X acting properly discontinuously and cocompactly.*

Then either Γ contains a free nonabelian subgroup, or else X is isometric to a Euclidean space and Γ is a Bieberbach group.

Proof. If X is not a Euclidean space, then X contains an $(n-1)$-simplex B_1 which bounds three n-simplices C_1^-, C_1^+ and C_1, where $n = \dim X \geq 1$. By Theorem 8.9, there is a Γ-compact flat F_1 containing C_1^- and C_1^+. In particular, there is $\phi_1 \in \Gamma$ translating F_1 in a direction perpendicular to B_1. Again by Theorem 8.9, there is a Γ-compact flat F containing C_1 and a $\phi \in \Gamma$ translating F in a direction perpendicular to B_1. Choose a point $x_1 \in B_1$ and let $\sigma_1 \ni x_1$ (respectively $\sigma \ni x_1$) be the geodesic in F_1 (respectively F) perpendicular to B_1. Then σ_1 is shifted by ϕ_1 and σ by ϕ.

Let $x_2 = \phi(x_1)$ and $B_2 = \phi(B_1)$. Then B_2 is an $(n-1)$-simplex which bounds at least three n-simplices C_2^-, C_2^+ and C_2, where C_2 is the last n-simplex through which σ passes before it meets B_2. By Theorem 8.9, there is a Γ-compact flat F_2 containing C_2^- and C_2^+. As above we conclude that the geodesic σ_2 in F_2 through x_2 and perpendicular to B_2 is shifted by an isometry $\phi_2 \in \Gamma$. Let $t_i > 0$ be the period of σ_i with respect to ϕ_i, that is, $\phi_i(\sigma_i(t)) = \sigma_i(t + t_i)$ for all $t \in \mathbb{R}, i = 1, 2$. Let U_i be the set of points x in X whose projection $P_i(x)$ onto σ_i is not in $\sigma_i((-t_i/2, t_i/2))$. Then

$$\phi_i^n(X \setminus U_i) \subset U_i \quad \text{for } n \neq 0,\ i = 1, 2. \tag{8.11}$$

Denote by $\bar{\sigma}$ the segment of σ between x_1 and x_2. By construction, the angles between the outgoing direction of $\bar{\sigma}$ at x_1 and the incoming and outgoing directions of σ_1 are π.

Similarly, the angles between the incoming direction of $\bar{\sigma}$ at x_2 and the incoming and outgoing directions of σ_2 are π. Therefore, each of the rays $\sigma_1((-\infty,0])$ and $\sigma_1([0,\infty))$ joined with $\bar{\sigma}$ and then extended by any of the two rays $\sigma_2((-\infty,0])$ or $\sigma_2([0,\infty))$ is a geodesic. We claim that

$$P_1(U_2) = x_1 \quad \text{and} \quad P_2(U_1) = x_2. \tag{8.12}$$

Suppose, for example, that x is a point with $p_1(x) = \sigma_1(t)$ and $t \leq -t_1/2$. Note that $y = p_1(x)$ is the unique point on σ_1 with $\angle_y(x,\sigma_1) \geq \pi/2$. Now let ω be the geodesic consisting of the concatenation of $\sigma_1((-\infty,0])$, $\bar{\sigma}$ and $\sigma_2([0,\infty))$. Since $t \leq -t_1/2$, we also have $\angle_y(x,\omega) \geq \pi/2$. Hence

$$\angle_z(x,\sigma_2) = \angle_z(x,\omega) < \pi/2$$

for all $z \in \sigma_2((0,\infty))$. Therefore $p_2(x) \notin \sigma_2((0,\infty))$. The other cases are treated similarly and (8.12) follows.

We conclude from (8.12) that $U_1 \cap U_2 = \emptyset$. Let x be a point on $\bar{\sigma}$. Then $p_1(x) = x_1$ and $p_2(x) = x_2$, hence $x \in X \setminus (U_1 \cup U_2)$.

Now consider any nontrivial reduced word w in ϕ_1 and ϕ_2. It follows from (8.11) that $w(x) \in U_i$ if w starts with a power of $\phi_i, i = 1,2$. Therefore $w(x) \neq x$, and hence $w \neq id$. Therefore ϕ_1 and ϕ_2 generate a free nonabelian subgroup of Γ. □

We now come to Theorem E of the Introduction.

8.13 Theorem. *Let (X,Γ) be a compact 2-dimensional boundaryless orbihedron with a piecewise smooth metric of nonpositive curvature.*

Then either Γ contains a free nonabelian subgroup or else X is isometric to the Euclidean plane and Γ is a Bieberbach group.

Proof. According to Theorem C, there are three cases to consider: If (X,Γ) has rank 1, then Γ contains a free nonabelian subgroup by Theorem 4.6. If X is a thick Euclidean building of type A_2, B_2 or G_2, then Γ contains a free nonabelian subgroup by Theorem 8.10. In the remaining case, X is the product of two trees T_1 and T_2. If X is not isometric to the Euclidean plane, then T_1 or T_2 has vertices with valency ≥ 3. Since the essential edges of $X = T_1 \times T_2$ are parallel and perpendicular to the factors, Γ preserves the product structure.

If both T_1 and T_2 have vertices of valency ≥ 3, declare their maximal edges (maximal arcs not containing vertices with valency ≥ 3) to have length 1. Then the barycentric subdivision of the unit squares in X is a triangulation of X which turns X into a Euclidean building of type B_2. Clearly, Γ acts by automorphisms on this building, and hence Γ contains a free nonabelian subgroup.

If one of the factors, say $T := T_1$, has vertices of valency ≥ 3 and the other is a line, $T_2 \cong \mathbb{R}$, then each $\phi \in \Gamma$ is of the form $\phi = (\psi,\tau)$, where ψ is an isometry of T and τ an isometry of $\mathbb{R}$. By passing to a subgroup of Γ of index 2, we may assume that each such τ is a translation. Since Γ is a cocompact and properly discontinuous group of automorphisms, the center of Γ consists of elements of the form $(1,\tau)$. Now let Γ' be the group of automorphisms ψ of T such that there is a translation τ of $\mathbb{R}$ with $(\psi,\tau) \in \Gamma$. Then Γ' is cocompact and finitely generated. Let J be a finite generating set.

We show now that Γ' is properly discontinuous, compare [Eb1, Lemma 5.1]. If ψ_n is a sequence in Γ' with $\psi_n \to id$, then the commutators $[\psi_n, \psi] \to id$ for any $\psi \in J$. Choose translations τ_n, τ of $\mathbb{R}$ such that $(\psi_n, \tau_n), (\psi, \tau) \in \Gamma$. Then

$$[(\psi_n, \tau_n), (\psi, \tau)] = ([\psi_n, \psi], 0) \to id.$$

Since Γ acts properly discontinuously, we conclude that $[\psi_n, \psi] = id$ for all n sufficiently large and any $\psi \in J$. Since J generates Γ', it follows that (ψ_n, τ_n) is in the center of Γ, and hence $\psi_n = id$ for all n sufficiently large. Hence Γ' is a properly discontinuous and cocompact group of automorphisms of T. Therefore Γ', and hence also Γ, contain a free nonabelian subgroup. $\square$

As another application of Theorem 8.9 we state the following generalization of Theorem 2 in [BaBu] to higher dimensions. The proof uses the arguments of [BaBu] and Theorem 8.9. The possibility of extending the arguments of [BaBu] to higher dimensions was indicated by M.Gromov (private communication), who also had a (different) approach for proving Theorem 8.9.

8.14 Theorem. *Let X be a thick Euclidean building, Γ a properly discontinuous and cocompact group of automorphisms of X and d a Γ-invariant metric of nonpositive curvature on X.*

Then, up to a Γ-equivariant homeomorphism and rescaling, d is the standard metric.

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