LU decomposition -- manual demonstration.
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LU decomposition, where L is a lower-triangular matrix with 1 as the diagonal elements and U is an upper-triangular matrix. Just as there are many combinations of $12=1\cdot 12=2\cdot 6=3\cdot 4=4\cdot 3=\ldots$, there are infinite number of combinations of $L\cdot U$. However, when the diagonal elements of L are fixed to be 1, the remaining elements are uniquely fixed.

$A = L\cdot U$ linear algebraic equation $A\cdot x = b \longrightarrow L\cdot U\cdot x = b \longrightarrow L\cdot y = b$ where $U\cdot x = y$

matrix inverse $A^{-1} = (L\cdot U)^{-1} = U^{-1}\cdot L^{-1}$

After LU decomposition, we obtain solution x in a two-step process

Step 0. $A=L\cdot U$
Step 1. Solve $L\cdot y=b \longrightarrow y=L^{-1}\cdot b$
Step 2. Solve $U\cdot x=y \longrightarrow x=U^{-1}\cdot y$

Example

$$A := \begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \qquad b := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$A = L\cdot U$

$$\begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

work on the 1st row of A

$A_{11} = 0 = 1\cdot U_{11} \longrightarrow U_{11} = A_{11} = 0$

$A_{12} = 1 = 1\cdot U_{12} \longrightarrow U_{12} = A_{12} = 1$

$A_{13} = 2 = 1\cdot U_{13} \longrightarrow U_{13} = A_{13} = 2$

$$\begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11}=0 & U_{12}=1 & U_{13}=2 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

work on the 2nd row of A

$A_{21} = 4 = L_{21}\cdot U_{11} \longrightarrow L_{21} = \dfrac{A_{21}}{U_{11}} = \dfrac{4}{0}$ ... divide by 0! ... We stop here!

$$\begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21}=\blacksquare & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11}=0 & U_{12}=1 & U_{13}=2 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

For each row, there is a step where we divide by the diagonal element of A. If any of the diagonal element of A is 0, LU decomposition does not exist. Since which equation comes first makes no difference in the solution of x, we swap equations, which is equivalent to swapping rows of both A and b.

Pivot. Examine column #1 of all the rows in A, the row with the largest element in this 1st column (in the absolute value sense) becomes the 1st row of the permutated matrix A'. Likewise swapping for b.
Examine column #2 of all the rows from row#2 to the last row in A, the row with the largest element in this 2nd column (in the absolute value sense) becomes the 2nd row of the permutated matrix A'. And so on...

$$A := \begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \text{ swap rows} \longrightarrow A' := \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \qquad b := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ swap rows} \longrightarrow b' := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A' = L \cdot U$$

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

If we work systematically from the first row of A', we can solve for unknown elements in L and U matrices sequentially, each time with only one unknown.

work on the 1st row of A'

$$A'_{11} = 4 = 1 \cdot U_{11} \longrightarrow U_{11} = A'_{11} = 4$$

$$A'_{12} = 1 = 1 \cdot U_{12} \longrightarrow U_{12} = A'_{12} = 1$$

$$A'_{13} = 0 = 1 \cdot U_{13} \longrightarrow U_{13} = A'_{13} = 0$$

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11}=4 & U_{12}=1 & U_{13}=0 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

work on the 2nd row of A'

$$A'_{21} = 1 = L_{21} \cdot U_{11} \longrightarrow L_{21} = \frac{A'_{21}}{U_{11}} = \frac{1}{4}$$

$$A'_{22} = 2 = L_{21} \cdot U_{12} + 1 \cdot U_{22} \longrightarrow U_{22} = A'_{22} - L_{21} \cdot U_{12} = 2 - \frac{1}{4} \cdot 1 = \frac{7}{4}$$

$$A'_{23} = 3 = L_{21} \cdot U_{13} + 1 \cdot U_{23} \longrightarrow U_{23} = A'_{23} - L_{21} \cdot U_{13} = 3 - \frac{1}{4} \cdot 0 = 3$$

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21}=\frac{1}{4} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11}=4 & U_{12}=1 & U_{13}=0 \\ 0 & U_{22}=\frac{7}{4} & U_{23}=3 \\ 0 & 0 & U_{33} \end{bmatrix}$$

work on the 3rd row of A'

$$A'_{31} = 0 = L_{31} \cdot U_{11} \longrightarrow L_{31} = \frac{A'_{31}}{U_{11}} = \frac{0}{4} = 0$$

$$A'_{32} = 1 = L_{31} \cdot U_{12} + L_{32} \cdot U_{22} \longrightarrow L_{32} = \frac{A'_{32} - L_{31} \cdot U_{12}}{U_{22}} = \frac{1 - 0 \cdot 1}{\frac{7}{4}} = \frac{4}{7}$$

$$A'_{33} = 2 = L_{31} \cdot U_{13} + L_{32} \cdot U_{23} + 1 \cdot U_{33} \longrightarrow U_{33} = A'_{33} - L_{31} \cdot U_{13} - L_{32} \cdot U_{23} = 2 - 0 \cdot 0 - \frac{4}{7} \cdot 3 = \frac{2}{7}$$

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} = \frac{1}{4} & 1 & 0 \\ L_{31} = 0 & L_{32} = \frac{4}{7} & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11} = 4 & U_{12} = 1 & U_{13} = 0 \\ 0 & U_{22} = \frac{7}{4} & U_{23} = 3 \\ 0 & 0 & U_{33} = \frac{2}{7} \end{bmatrix}$$

Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{7} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{7}{4} & 3 \\ 0 & 0 & \frac{2}{7} \end{bmatrix}$$

check:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{7} & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{7}{4} & 3 \\ 0 & 0 & \frac{2}{7} \end{bmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

Step 1. Solve $L \cdot y = b' \longrightarrow y = L^{-1} \cdot b'$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{7} & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = b' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$1 \cdot y_1 = b'_1 \longrightarrow y_1 = b'_1 = 1$$

$$L_{21} \cdot y_1 + 1 \cdot y_2 = b'_2 \longrightarrow y_2 = b'_2 - L_{21} \cdot y_1 = 0 - \frac{1}{4} \cdot 1 = -\frac{1}{4}$$

$$L_{31} \cdot y_1 + L_{32} \cdot y_2 + 1 \cdot y_3 = b'_3 \longrightarrow y_3 = b'_3 - L_{31} \cdot y_1 - L_{32} \cdot y_2 = 0 - 0 \cdot 1 - \frac{4}{7} \cdot \left(-\frac{1}{4}\right) = \frac{1}{7}$$

Step 2. Solve $U \cdot x = y \longrightarrow x = U^{-1} \cdot y$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{7}{4} & 3 \\ 0 & 0 & \frac{2}{7} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = y = \begin{bmatrix} 1 \\ -\frac{1}{4} \\ \frac{1}{7} \end{bmatrix}$$

$$U_{33} \cdot x_3 = y_3 \longrightarrow x_3 = \frac{y_3}{U_{33}} = \frac{\frac{1}{7}}{\frac{2}{7}} = \frac{1}{2}$$

$$U_{22} \cdot x_2 + U_{23} \cdot x_3 = y_2 \longrightarrow x_2 = \frac{y_2 - U_{23} \cdot x_3}{U_{22}} = \frac{-\frac{1}{4} - 3 \cdot \frac{1}{2}}{\frac{7}{4}} = -1$$

$$U_{11} \cdot x_1 + U_{12} \cdot x_2 + U_{13} \cdot x_3 = y_1 \longrightarrow x_1 = \frac{y_1 - U_{12} \cdot x_2 - U_{13} \cdot x_3}{U_{11}} = \frac{1 - 1 \cdot (-1) - 0 \cdot \frac{1}{2}}{4} = \frac{1}{2}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad x := \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}$$

check:

$$A \cdot x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longleftarrow \text{compare} \longrightarrow b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A' \cdot x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longleftarrow \text{compare} \longrightarrow b' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Swapping rows of A does not affect the answer x, as long as rows of b are also similarly swapped.

Mathcad's lu function returns 3 matrices: P, L, U such that P·A=L·U.
P is a permutation matrix that has "1" occupying some elements $P_{i,j}$ that signifies the raw swapping operation from row j to row i.

$PLU := lu(A)$

$$PLU = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0.25 & 1 & 0 & 0 & 1.75 & 3 \\ 1 & 0 & 0 & 0 & 0.571 & 1 & 0 & 0 & 0.286 \end{pmatrix}$$

$P := submatrix(PLU,1,3,1,3)$
$L := submatrix(PLU,1,3,4,6)$
$U := submatrix(PLU,1,3,7,9)$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0.571 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 1.75 & 3 \\ 0 & 0 & 0.286 \end{pmatrix}$$

Pre-multiplication by a permutation matrix = **row swapping**
The 1st row of P has P_{12}=1 ⟶ 2nd row in A goes into 1st row in A'.
The 2nd row of P has P_{23}=1 ⟶ 3rd row in A goes into 2nd row in A'.
The 3rd row of P has P_{31}=1 ⟶ 1st row in A goes into 3rd row in A'.
Thus, the permutated matrix A' has: row 2 ⟶ row 3 ⟶ row 1 of A.

check

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow P \cdot A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \longleftarrow \text{compare} \longrightarrow L \cdot U = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

P is orthonormal $P \cdot P^T = P^T \cdot P = I$ $\quad P^{-1} = P^T$

$$P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad P^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Applying P^T to the permutated matrix A' reverses the orignal permutation and yields back the original matrix A.

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.143 & -0.571 & 1 \end{pmatrix} \quad U^{-1} = \begin{pmatrix} 0.25 & -0.143 & 1.5 \\ 0 & 0.571 & -6 \\ 0 & 0 & 3.5 \end{pmatrix}$$

$$A'^{-1} = \begin{pmatrix} 0.5 & -1 & 1.5 \\ -1 & 4 & -6 \\ 0.5 & -2 & 3.5 \end{pmatrix} \longleftarrow \text{compare} \longrightarrow U^{-1} \cdot L^{-1} = \begin{pmatrix} 0.5 & -1 & 1.5 \\ -1 & 4 & -6 \\ 0.5 & -2 & 3.5 \end{pmatrix}$$

Post-multiplication by a permutation matrix = **column swapping**
In the equation below, $P^{-1}=P^T$ is also a permutation matrix. Post-multiplying of A^{-1} by $P^{-1}=P^T$ has the following effect:
The 1st column of P^{-1} has $(P^{-1})_{21}$=1 ⟶ 2nd column in A^{-1} goes into 1st column in A'^{-1}.
The 2nd column of P^{-1} has $(P^{-1})_{32}$=1 ⟶ 3rd column in A^{-1} goes into 2nd column in A'^{-1}.
The 3rd column of P^{-1} has $(P^{-1})_{13}$=1 ⟶ 1st column in A^{-1} goes into 3rd column in A'^{-1}.
Thus, the permutated matrix A'^{-1} has: column 2 ⟶ column 3 ⟶ column 1 of A^{-1}.
Swapping rows of A results in **swapping columns** of A^{-1} in the same order.

$A^{-1} = \begin{pmatrix} 1.5 & 0.5 & -1 \\ -6 & -1 & 4 \\ 3.5 & 0.5 & -2 \end{pmatrix}$ ⟵ compare ⟶ $A^{-1} \cdot P^{-1} = \begin{pmatrix} 0.5 & -1 & 1.5 \\ -1 & 4 & -6 \\ 0.5 & -2 & 3.5 \end{pmatrix}$ ⟷ $A'^{-1} = \begin{pmatrix} 0.5 & -1 & 1.5 \\ -1 & 4 & -6 \\ 0.5 & -2 & 3.5 \end{pmatrix}$

Effect of swapping rows on matrix inverse

$$I = (P \cdot A) \cdot (P \cdot A)^{-1} = (P \cdot A) \cdot \left(A^{-1} \cdot P^{-1}\right) = (P \cdot A) \cdot \left(A^{-1} \cdot P^{T}\right)$$

$$A'^{-1} = A^{-1} \cdot P^{-1} = A^{-1} \cdot P^{T}$$

$$A^{-1} = A'^{-1} \cdot P$$

Post-multiplication by a permutation matrix = **column swapping**

In the equation above, post-multiplying of A'^{-1} by P has the following effect:

The 1st column of P has $(P)_{31}=1$ ⟶ 3rd column in A'^{-1} goes into 1st column in A^{-1}.

The 2nd column of P has $(P)_{12}=1$ ⟶ 1st column in A'^{-1} goes into 2nd column in A^{-1}.

The 3rd column of P has $(P)_{23}=1$ ⟶ 2nd column in A'^{-1} goes into 3rd column in A^{-1}.

Thus, the permutated matrix A'^{-1} has: column 2 ⟶ column 3 ⟶ column 1 of A^{-1}.

From A'^{-1} to A^{-1}, **swap columns** of A'^{-1} in a **reverse** order.

Post-multiplication by a permutation matrix = **column** swapping

In the equation below, post-multiplying of A by P has the following effect:

The 1st column of P has $P_{31}=1$ ⟶ 3rd column in A goes into 1st column in A''.

The 2nd column of P has $P_{12}=1$ ⟶ 1st column in A goes into 2nd column in A''.

The 3rd column of P has $P_{23}=1$ ⟶ 2nd column in A goes into 3rd column in A''.

Thus, the permutated matrix A'' has: column 3 ⟶ column 1 ⟶ column 2 of A.

$A = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ ⟶ $A'' := A \cdot P$ $A'' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

Gaussian Elimination & LU Decomposition. Let us illustrate with the same matrix A and vector b as before.

$$A := \begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \qquad b := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Step 0. Augment matrix A and vector b

$$Ab := \text{augment}(A, b) \qquad Ab = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 4 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

We represent the steps Gaussian elimination takes in manipulating the elements in the augmented matrix Ab by pre-multiplying with a square matrix, which acts as an operator that operates on the second matrix. Pivoting: swap 1st & 2nd eqn, because eqn (1.2) has the largest leading coefficient:

$$P_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A'b' := P_1 \cdot Ab \qquad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix} \begin{matrix} (1.2) \\ (1.1) \\ (1.3) \end{matrix}$$

* (1.2) by 0/4 & subtract it from (1.1) ⟶ (2.2)
* (1.2) by 1/4 & subtract it from (1.3) ⟶ (2.3)

$$G_1 := \begin{bmatrix} 1 & 0 & 0 \\ -\frac{0}{4} & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix}$$

⟵ "1" in the diagonal position for the 1st row of G_1 means just transcribe the 1st row of A'b' and do nothing.
⟵ "-0/4" means subtract 0/4 of 1st row of A'b', and "1" means add 1x of 2nd row of A'b'.
⟵ "-1/4" means subtract 1/4 of 1st row of A'b', and "1" means add 1x of 3rd row of A'b'.

$$A'b' := G_1 \cdot A'b' \qquad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1.75 & 3 & -0.25 \end{pmatrix} \begin{matrix} (2.1) \\ (2.2) \\ (2.3) \end{matrix}$$

Pivoting: swap 2nd & 3rd eqn:

$$P_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad A'b' := P_2 \cdot A'b' \qquad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 0 & 1.75 & 3 & -0.25 \\ 0 & 1 & 2 & 0 \end{pmatrix} \begin{matrix} (2.1) \\ (2.3) \\ (2.2) \end{matrix}$$

* (2.3) by 1/(7/4) & subtract it from (2.2) ⟶ (3.3)

$$G_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{1.75} & 1 \end{bmatrix} \qquad A'b' := G_2 \cdot A'b' \qquad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 0 & 1.75 & 3 & -0.25 \\ 0 & 0 & 0.286 & 0.143 \end{pmatrix} \begin{matrix} (2.1) & (3.1) \\ (2.3) & (3.2) \\ (2.2) & (3.3) \end{matrix}$$

Below is a minor variation of the above steps where we perform all the pivoting first, rather than pivoting as we go in each step. A combination of two sequential swapping steps is equivalent to pre-multiplying the augmented matrix Ab by P, which does multiple swappings in one sweep.

$$P := P_2 \cdot P_1 \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad A'b' := P \cdot Ab \qquad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \begin{matrix} (1.2) \\ (1.3) \\ (1.1) \end{matrix}$$

* (1.2) by 1/4 & subtract it from (1.3) ⟶ (2.2)
* (1.2) by 0/4 & subtract it from (1.1) ⟶ (2.3)

$$G_1 := \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{0}{4} & 0 & 1 \end{bmatrix} \qquad A'b' := G_1 \cdot A'b' \qquad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 0 & 1.75 & 3 & -0.25 \\ 0 & 1 & 2 & 0 \end{pmatrix} \begin{matrix} (2.1) \\ (2.2) \\ (2.3) \end{matrix}$$

* (2.2) by 1/(7/4) & subtract it from (2.3) ⟶ (3.3)

$$G_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{1.75} & 1 \end{bmatrix} \qquad A'b' := G_2 \cdot A'b' \qquad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 0 & 1.75 & 3 & -0.25 \\ 0 & 0 & 0.286 & 0.143 \end{pmatrix} \begin{matrix} (3.1) \\ (3.2) \\ (3.3) \end{matrix}$$

We combine the two sequential Gaussian elimination steps G_1 & G_2 into an equivalent one single operation G:

$$G := G_2 \cdot G_1 \qquad G = \begin{pmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.143 & -0.571 & 1 \end{pmatrix} \qquad A'b' := G \cdot P \cdot Ab \qquad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 0 & 1.75 & 3 & -0.25 \\ 0 & 0 & 0.286 & 0.143 \end{pmatrix}$$

The following play on math shows that since the "A'"matrix in A'b is upper triangular, the inverse of G is lower triangular and this is the L matrix. Thus, the lower triangular matrix L summarizes all the individual forward elimination steps taken during Gaussian elimination leading up to an upper triangular form, and Gaussian elimination is directly related to LU decomposition.

$$A'b' = G \cdot P \cdot Ab$$

$$G^{-1} \cdot A'b' = P \cdot Ab \qquad A' := \text{submatrix}(A'b', 1, 3, 1, 3) \qquad A' = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 1.75 & 3 \\ 0 & 0 & 0.286 \end{pmatrix}$$

$$L \cdot A'b' = P \cdot Ab$$

$$L := G^{-1} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0.571 & 1 \end{pmatrix} \qquad \text{and} \qquad U = A'$$

Check:

$$L \cdot A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \longleftarrow \text{compare} \longrightarrow P \cdot Ab = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

$$L \cdot A' = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \longleftarrow \text{compare} \longrightarrow P \cdot A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$